HWRS 505: Vadose Zone Hydrology

Lecture 2

8/24/2023

Today:

1. Review: Steady-state saturated flow
2. Derive permeability from Hagen-Poiseuille flow

Steady-state saturated flow

Review of Lecture 1

- ❖ Vadose zone (Overview)
  - Conceptual picture
  - Societal impacts
  - It's role in the global hydrological and carbon cycles, and the global surface energy balances
- ❖ Steady-state saturated flow
  - Energy potential; hydraulic head
  - Darcy's law; saturated hydraulic conductivity; permeability

If not, $\int_{p_0}^{p_1} \frac{1}{\rho(p)} dp$ (Instead of $\frac{1}{\rho}(p_1 - p_0)$)

Assumptions:
1° volume of the fluid parcel does not change
2° kinetic energy is neglected. $\frac{1}{2}(v_1^2 - v_0^2) \approx 0$

K $[L/T]$ k $[L^2]$

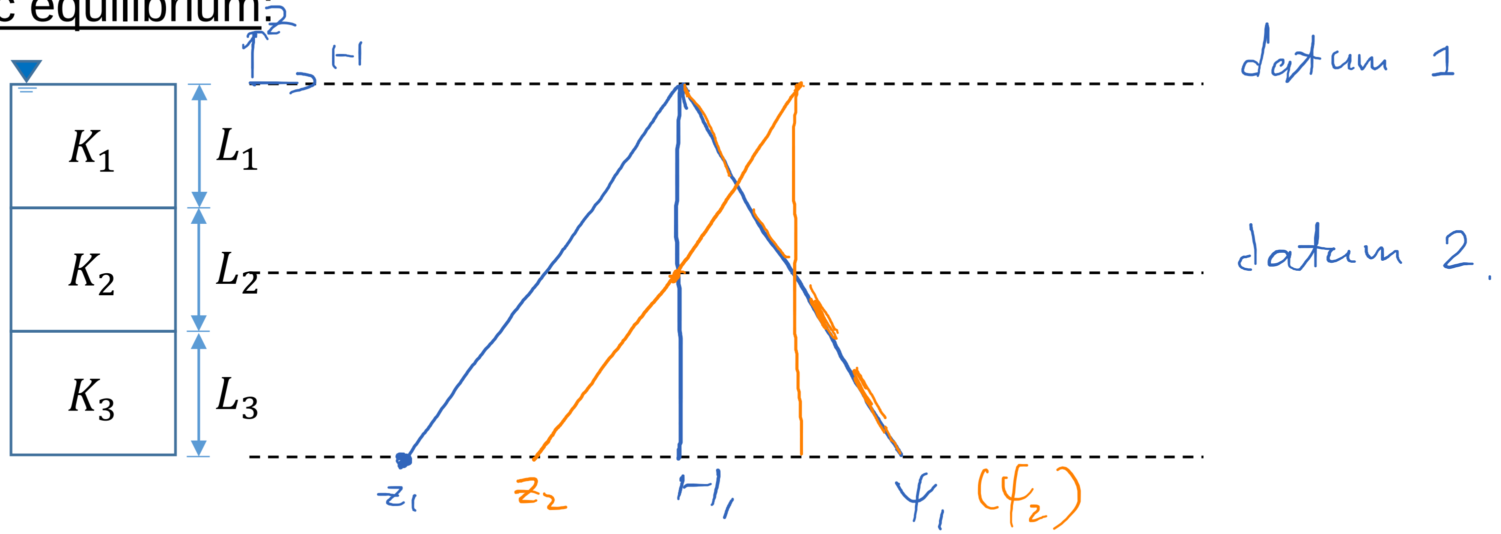
$$q = -K \frac{H_2 - H_1}{L}$$

$$q = -K \frac{dH}{dx} \quad \text{(differential form in 1D)}$$

$$\underset{\sim}{q} = -\underset{\approx}{K} \underset{\sim}{\nabla} H \quad \text{(differential form in 3D)}$$

Steady-state saturated flow

Static equilibrium:



$$H_1 = z_1 + \psi_1$$
$$H_2 = z_2 + \psi_2$$

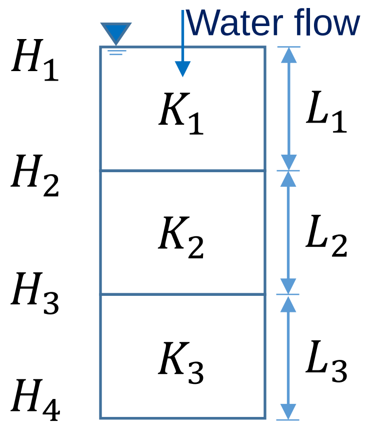
1°. Hydraulic head is constant in space.

2°. Water pressure head remains the same for different datum.

3°. The solution does not change w/ K_1, K_2, K_3.

Steady-state saturated flow

Steady-state flow:



(1) Homogeneous: $K_1 = K_2 = K_3$
(2) Heterogeneous: $K_1 \neq K_2 \neq K_3$

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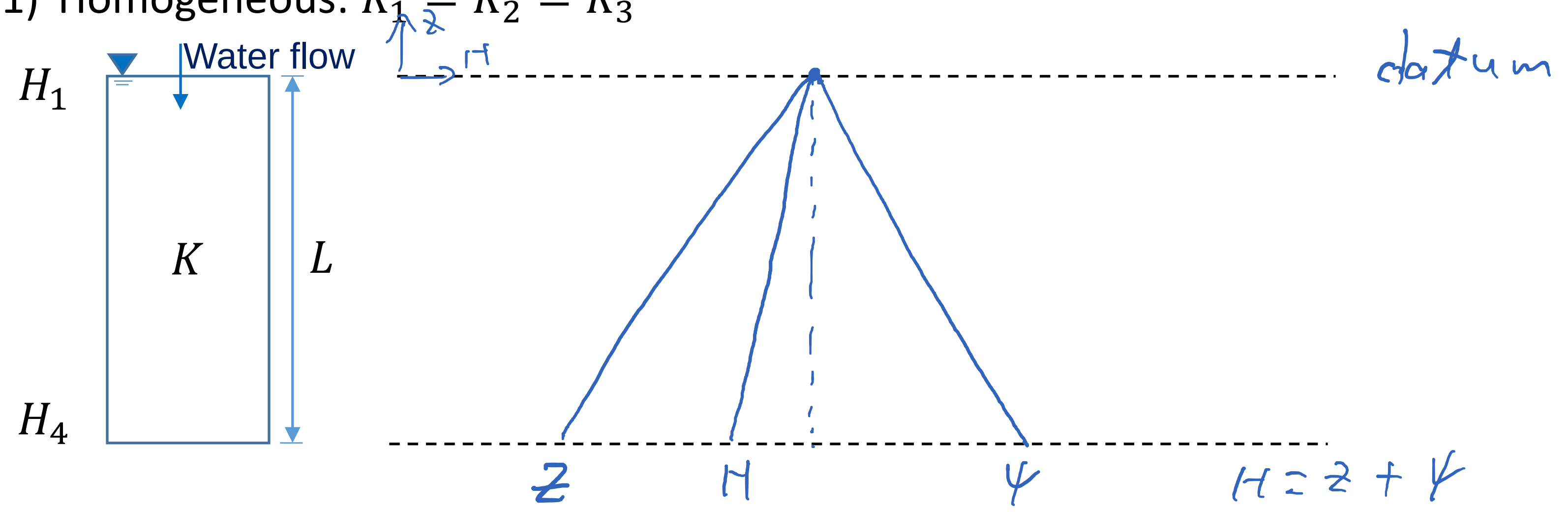


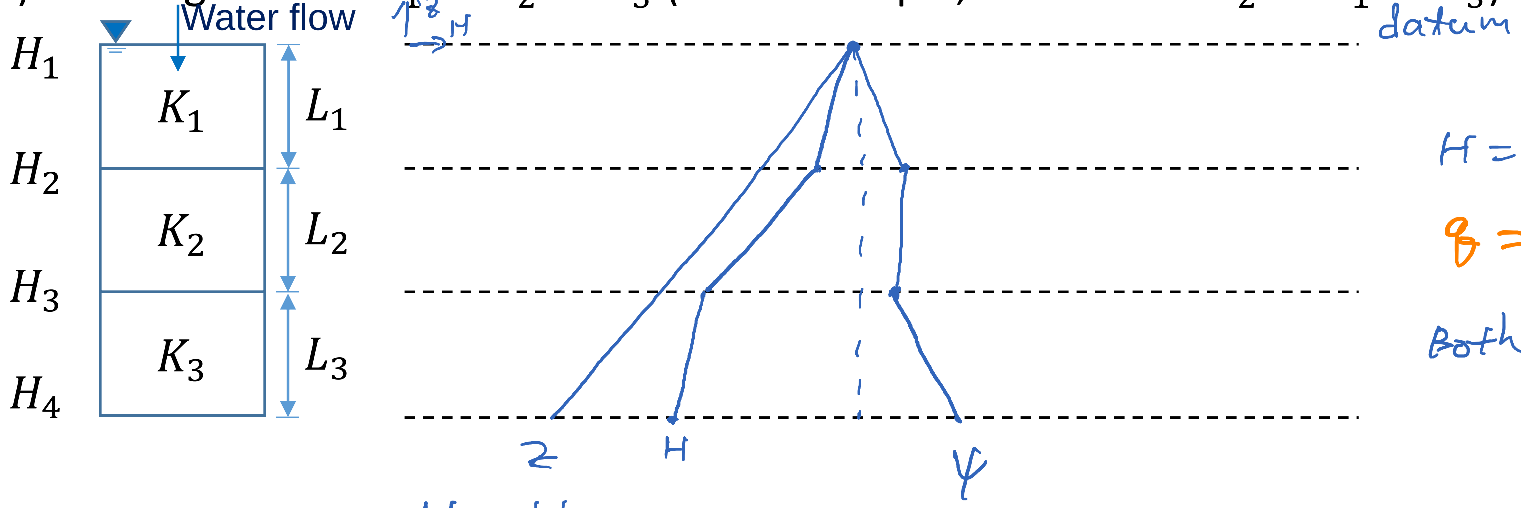
$H = z + \psi$

$q = -K \frac{dH}{dz}$

H decreases linearly along the flow direction.

Steady-state saturated flow

(2) Heterogeneous: $K_1 \neq K_2 \neq K_3$ (as an example, we assume $K_2 < K_1 = K_3$)



$$H = z + \psi$$

$$q = -K \frac{dH}{dz}$$

Both H and ψ decrease piecewise linearly along the flow direction.

$$Q = -K_{eq} A \frac{H_4 - H_1}{L_1 + L_2 + L_3}$$

$$Q_1 = -K_1 A \frac{H_2 - H_1}{L_1}$$

$$Q_2 = -K_2 A \frac{H_3 - H_2}{L_2}$$

$$Q_3 = -K_3 A \frac{H_4 - H_3}{L_3}$$

$$\Rightarrow H_4 - H_1 = -\frac{Q}{A}\left(\frac{L_1}{K_1} + \frac{L_2}{K_2} + \frac{L_3}{K_3}\right)$$

$$\Rightarrow K_{eq} = \frac{L_1 + L_2 + L_3}{\frac{L_1}{K_1} + \frac{L_2}{K_2} + \frac{L_3}{K_3}} = \frac{\sum_i L_i}{\sum_i \frac{L_i}{K_i}}$$

Steady-state saturated flow

HWRS 505
Bo Guo
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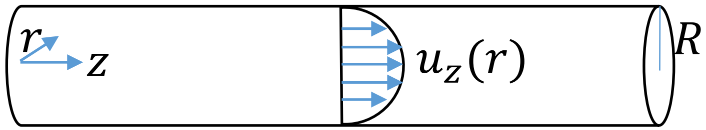
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}} + \cancel{\frac{\partial^2}{\partial z^2}}$$

Permeability $[L^2]$

"Hagen-Poiseuille" flow



Note for the Laplacian: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$

Notes for Darcy's Law:

$$q = -K\nabla H = -K\nabla\psi$$
$$= -\frac{k\rho g}{\mu}\nabla\psi$$
$$= -\frac{k}{\mu}\nabla p$$

$K = \frac{k\rho g}{\mu}$

$p = \rho g \psi$

Navier-Stokes Equation

$$\rho\cancel{\frac{Du_z}{Dt}}^{\,0} = -\frac{\partial p}{\partial z} + \cancel{f_z}^{\,0} + \mu\nabla^2 u_z$$

(steady-state Low Re#)

(neglect body force for simplicity)

$$\Rightarrow \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{\partial p}{\partial z}$$
$$\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{r}{\mu}\frac{\partial p}{\partial z}$$
$$r\frac{\partial u_z}{\partial r} = \frac{r^2}{2\mu}\frac{\partial p}{\partial z} + C_1$$
$$\frac{\partial u_z}{\partial r} = \frac{r}{2\mu}\frac{\partial p}{\partial z} + \frac{1}{r}C_1$$

$$\Rightarrow u_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + C_1\ln r + C_2$$

BC:

$$u_z|_{r=R} = 0 \Rightarrow C_2 = -\frac{R^2}{4\mu}\frac{\partial p}{\partial r}$$
$$\frac{\partial u_z}{\partial r}|_{r=0} = 0 \Rightarrow C_1 = 0$$

$$\Rightarrow u_z = \frac{1}{4\mu}\frac{\partial p}{\partial z}(r^2 - R^2)$$

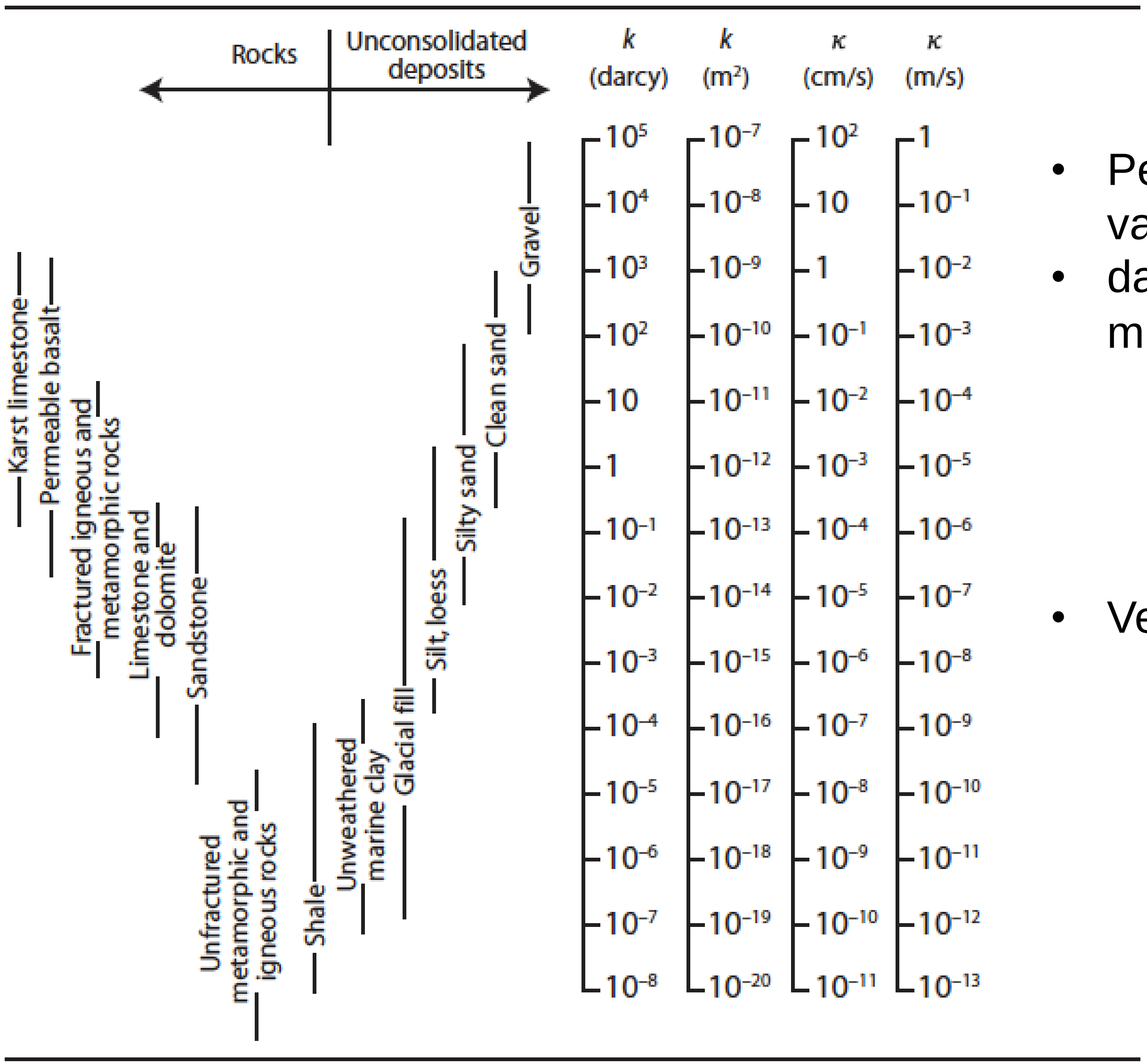
$$Q = \int_0^R 2\pi r u_z\,\mathrm{d}r = \int_0^R \frac{\pi}{2\mu}\frac{\partial p}{\partial z}(r^3 - R^3)\,\mathrm{d}r$$

$$\Rightarrow q = \frac{Q}{A} = -\frac{1}{8\mu}R^2\frac{\partial p}{\partial z}$$

$$\Rightarrow k = R^2/8 \qquad [L^2]$$

$\boxed{\text{=> The permeability of the tube is } k = R^2/8}$

Steady-state saturated flow



- Permeability and conductivity values for various soil and rock media
- darcy (d) = $9.869233 \times 10^{-13}\ \mathrm{m}^2 \approx 1\ \mu\mathrm{m}^2$. millidarcy (md) = 0.001 darcy.

  A porous medium with a permeability of 1 darcy permits a flow of 1 cm³/s of a fluid with viscosity 1 cP (1 mPa·s) under a pressure gradient of 1 atm/cm acting across an area of 1 cm².

  9.869233=1/1.013250. 1 atm = 1.013250 x 10^5 Pa

- Velocity of groundwater flow $\sim 1\ \mathrm{m/day}$

(Freeze and Cherry, 1979)