HWRS 505: Vadose Zone Hydrology

Lecture 3

8/29/2023

Today:

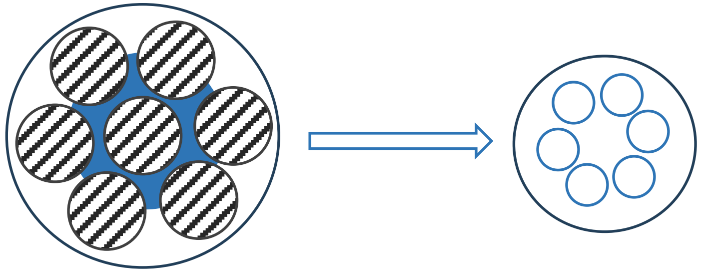
1. Derive the 3D groundwater flow equation
2. Review solute transport under saturated flow

Steady-state saturated flow

Review of Lecture 2

❖ Steady-state saturated flow

- Boundary conditions -> Distribution of hydraulic head and pressure head.
  - ✓ The definition of total hydraulic head ($H = z + \psi$)
  - ✓ Darcy's law ($q = -K\frac{\mathrm{d}H}{\mathrm{dz}}$)
- Derivation of effective conductivity for a layered system.
- Derivation of permeability for a tube based on the Hagen-Poiseuille flow (simplified N-S equation).
  - ✓ Permeability ~ R^2
  - ✓ The origin of Darcy's law is N-S equation
  - ✓ Permeability for a bundle of tubes of uniform size?



n tubes
k_{ST}
ϕ

$$\bar{q} = \frac{nq\pi R^2}{n\pi R^2/\phi} = \phi q = -\phi\frac{k_{ST}}{\mu}\frac{\partial p}{\partial x}$$

$$\bar{q} = -\frac{k_{BoT}}{\mu}\frac{\partial p}{\partial x}$$

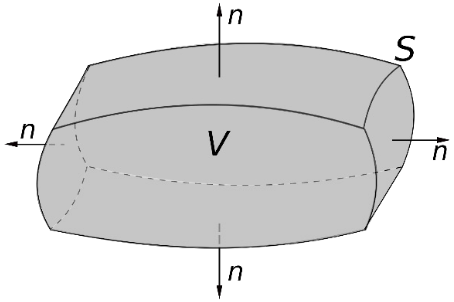
$$k_{BoT} = \phi k_{ST}$$

Steady-state saturated flow

<u>Derive the flow equation</u>:

Recall: Divergence theorem

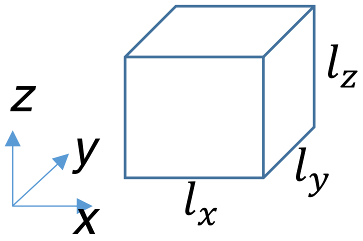
$$\int_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{q}\,\mathrm{d}V$$



=> Divergence theorem converts a surface integral to a volume integral.

<u>Mass conservation</u>: Change of mass storage = mass in – mass out.

porosity.



Rate of mass change: $\frac{d}{dt}\int_0^{l_x}\int_0^{l_y}\int_0^{l_z}\rho(x,y,z)\,\phi(x,y,z)\,dx\,dy\,dz = \frac{d}{dt}\int_\Omega \rho\phi\,dV$

Net fluxes: $\int_0^{l_y}\int_0^{l_z} q_x\big|_{x=0}\,dy\,dz - \int_0^{l_y}\int_0^{l_z} q_x\big|_{x=l_x}\,dy\,dz$

$+ \int_0^{l_x}\int_0^{l_z} q_y\big|_{y=0}\,dx\,dz - \int_0^{l_x}\int_0^{l_z} q_y\big|_{y=l_y}\,dx\,dz$

$+ \int_0^{l_x}\int_0^{l_y} q_z\big|_{z=0}\,dx\,dy - \int_0^{l_x}\int_0^{l_y} q_z\big|_{z=l_z}\,dx\,dy$

Steady-state saturated flow

$$\Rightarrow -\int_{\partial\Omega} \underset{\sim}{q} \cdot \underset{\sim}{n} \, ds = -\int_{\Omega} \underset{\sim}{\nabla} \cdot \underset{\sim}{q} \, dV$$ Divergence theorem

Thus, $$\frac{d}{dt}\int_{\Omega} \rho\phi \, dV = -\int_{\Omega} \underset{\sim}{\nabla} \cdot \underset{\sim}{q} \, dV$$

$\Omega \neq \Omega(t)$

$$\Rightarrow \int_{\Omega} \frac{d}{dt}(\rho\phi) \, dV = -\int_{\Omega} \underset{\sim}{\nabla} \cdot \underset{\sim}{q} \, dV$$

$$\Rightarrow \int_{\Omega} \left[\frac{d}{dt}(\rho\phi) + \underset{\sim}{\nabla} \cdot \underset{\sim}{q}\right] dV = 0$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial}{\partial t}(\rho\phi) + \underset{\sim}{\nabla} \cdot \underset{\sim}{q}\right] dV = 0$$

Integral $=0$, Ω is arbitrary $\Rightarrow$ Integrand $=0$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\rho\phi) + \underset{\sim}{\nabla} \cdot \underset{\sim}{q} = 0$$

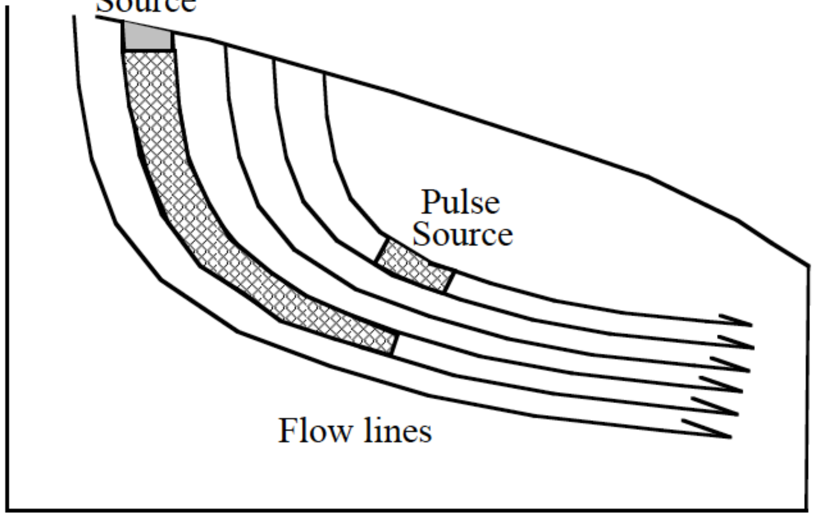
Darcy's law: $\underset{\sim}{q} = -\rho \underset{\approx}{K} \underset{\sim}{\nabla} H$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\rho\phi) + \underset{\sim}{\nabla} \cdot \left(-\rho \underset{\approx}{K} \underset{\sim}{\nabla} H\right) = 0}$$

General 3D flow equation for a ~~W~~ water saturated porous medium.

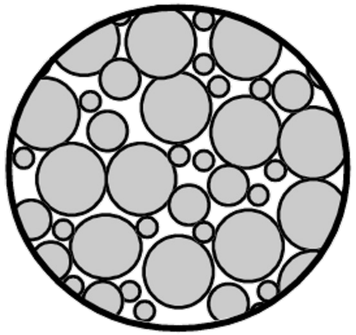
Solute transport under saturated flow

<u>Advection</u>: The solute particles move along streamlines with a velocity equal to the groundwater velocity



$$q = Q/A = -\underset{\sim}{K} \underset{\sim}{\nabla} H$$

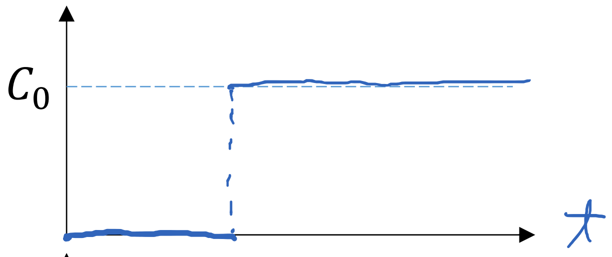
$$v = q/\phi$$



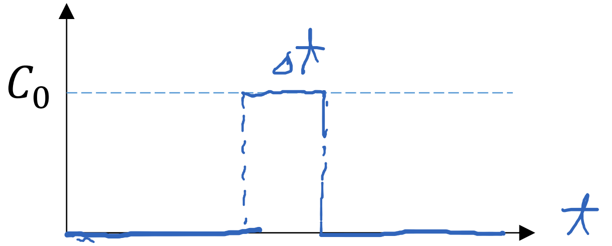
Porewater velocity is greater than the volumetric Darcy flux

Breakthrough curve (BTC)

Continuous source:



Pulse source:



Solute transport under saturated flow

Molecular diffusion: Solute particles move due to random molecular motion.

Free water: Recall Fick's Law

$$\boldsymbol{q}_c = -D_0 \boldsymbol{\nabla} \mathrm{C}.$$

Porous medium:

$$\boldsymbol{q}_c = -\phi D_d \boldsymbol{\nabla} \mathrm{C}.$$

$D_d = D_0 w$ is the effective diffusion coefficient, and w $(0 < w < 1)$ is the tortuosity factor. (Note: tortuosity vs. tortuosity factor)

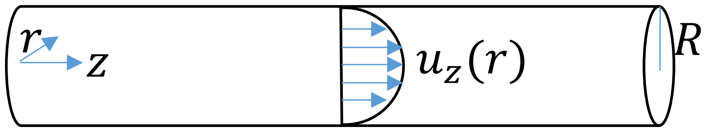
Mechanical dispersion: Mixing due to local-scale variations in groundwater velocity

Example: Dispersion in a capillary tube [G.I. Taylor (1953) and R. Aris (1956)]

Solute transport under saturated flow

Taylor-Aris Dispersion: Dispersion in a capillary tube [G.I. Taylor (1953) and R. Aris (1956)]

Solute transport in "Hagen-Poiseuille" flow



Governing equation for solute transport in the tube:

$$\frac{\partial C}{\partial t} + 2u\left[1 - \left(\frac{r}{R}\right)^2\right]\frac{\partial C}{\partial x} - D_0\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}\right) = 0$$

At sufficiently long times (derivation via the perturbation method)

$$\frac{\partial \bar{C}}{\partial t} + \bar{u}\frac{\partial \bar{C}}{\partial x} - D_L\frac{\partial^2 \bar{C}}{\partial x^2} = 0, \qquad D_L = D_0 + \frac{R^2\bar{u}^2}{48D_0}$$

$\bar{u}$: mean velocity in the tube.

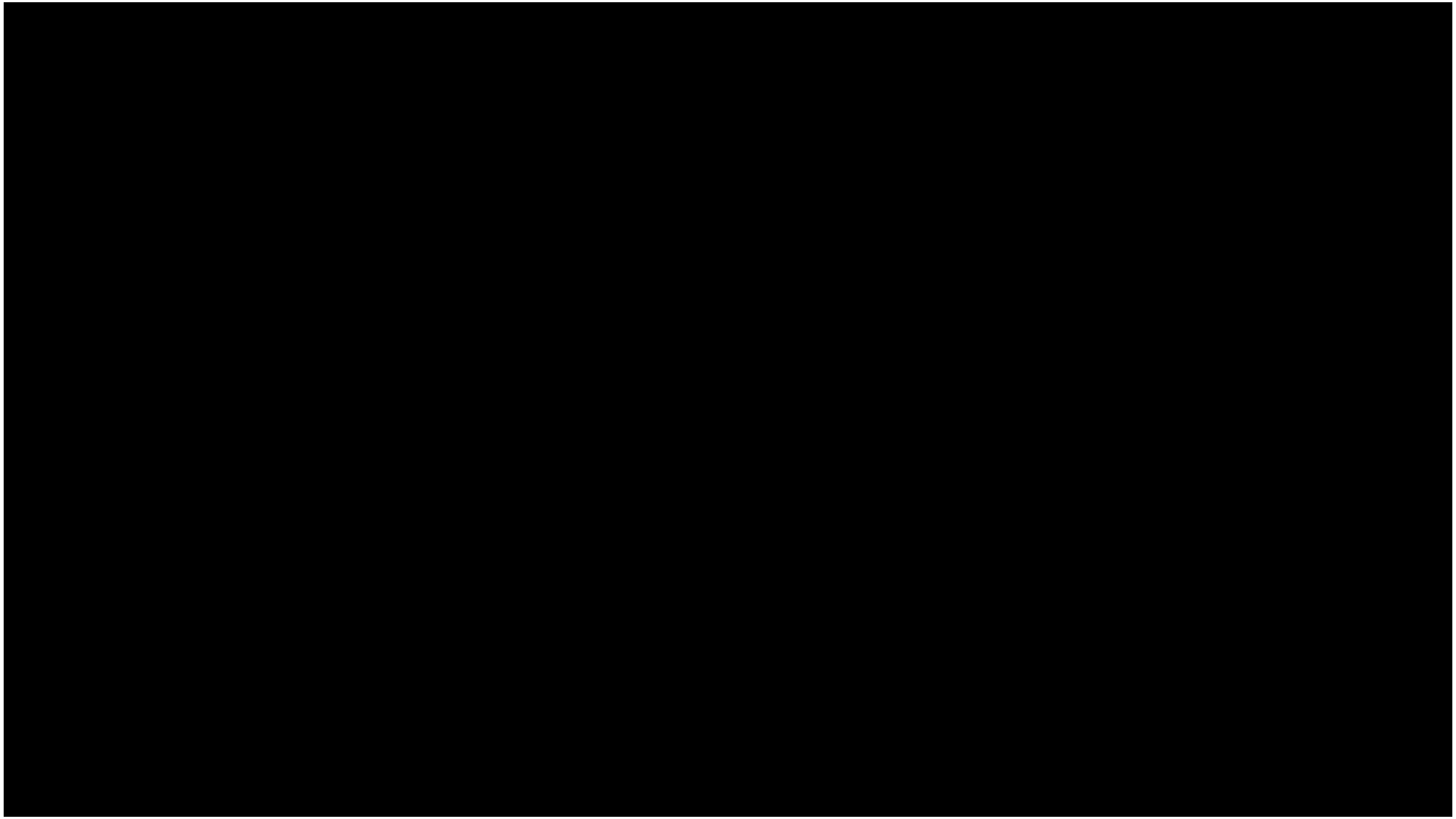
C_m: mean concentration in the tube.

$$C_m = \frac{\int_0^{2\pi}\int_0^R C(r,x)\, r\,\mathrm{d}r\,\mathrm{d}\theta}{\int_0^{2\pi}\int_0^R r\,\mathrm{d}r\,\mathrm{d}\theta} = \frac{2}{R^2}\int_0^R C\, r\,\mathrm{d}r$$

Insights: The average solute concentration spreads out by a dispersion process (radial diffusion and axial advection) and follows Fickian diffusion. The effective diffusivity is not the molecular diffusivity D_0, rather, it is a quadratic function of the mean velocity.

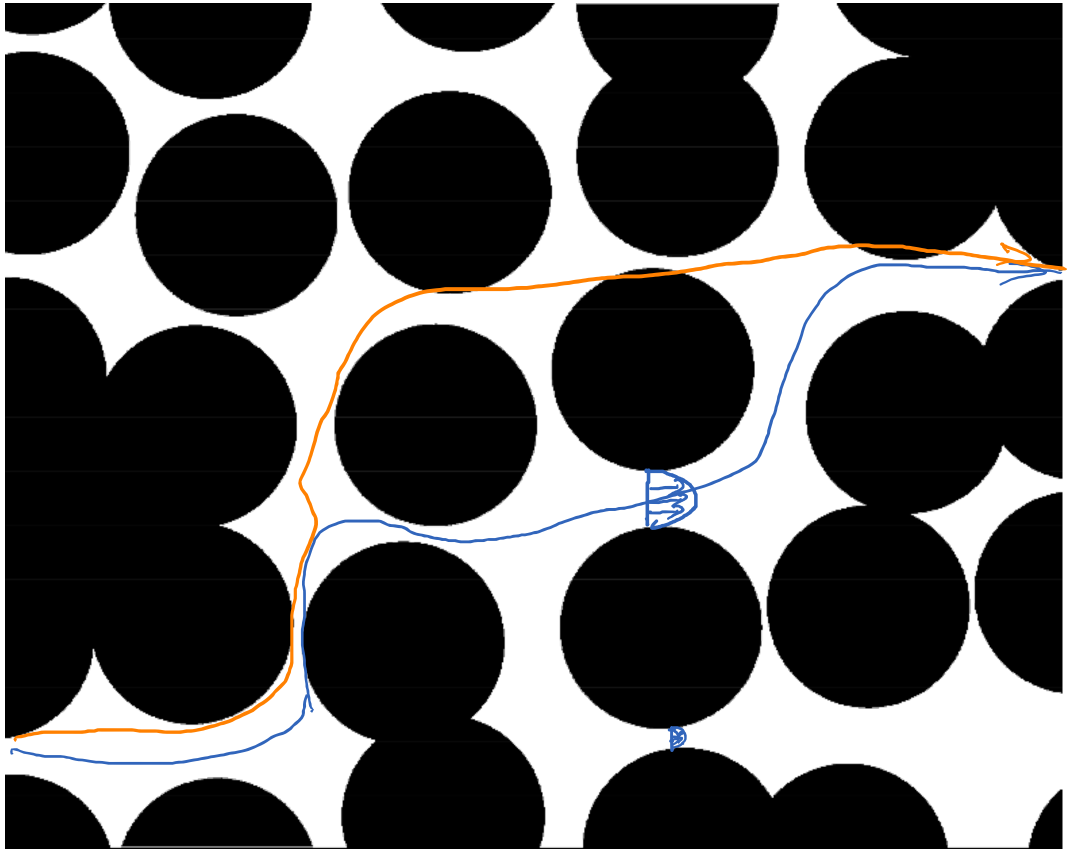
Solute transport under saturated flow

Illustrative numerical simulations of Taylor-Aris dispersion (https://youtu.be/toC4RM_aUS4)



Solute transport under saturated flow

Taylor-Aris Dispersion: Spatial variation of velocity leads to dispersion of solutes

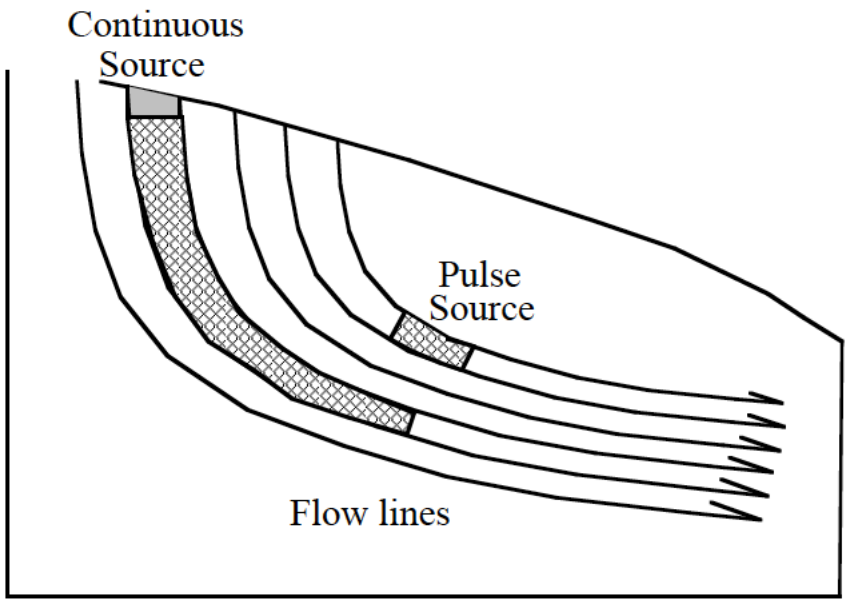


Spatial variation of velocity in a porous medium:

(1) Velocity profile through a pore throat (related to T-A dispersion)
(2) Velocity variation among pore throats
(3) Tortuosity

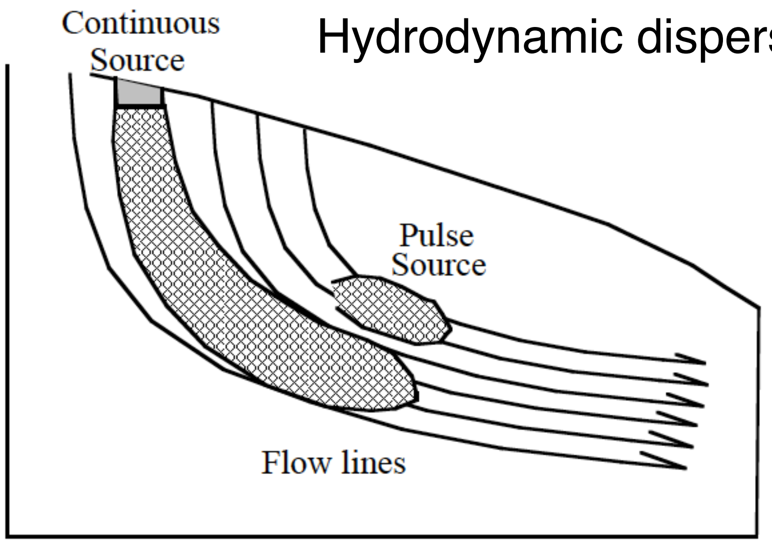
Solute transport under saturated flow

Pure advection



Advection + Mechanical dispersion + Molecular diffusion

Hydrodynamic dispersion



A widely used theory for describing solute flux in saturated porous media [J. Bear (1961)]

$$\boldsymbol{q}_c = \phi(\boldsymbol{v}C - \boldsymbol{D}\nabla C),$$

- $\boldsymbol{q}_c$ is the solute flux vector.
- $\boldsymbol{v}$ is the velocity field.
- $\boldsymbol{D}$ is a tensor that depends on the velocity field and molecular diffusion in the porous medium.

For a special case in 1D flow:

$$D_{xx} = \alpha_L v_x + D_d$$

$$D_{yy} = D_{zz} = \alpha_T v_x + D_d$$

- α_L and α_T are longitudinal and transverse dispersivities, respectively.