HWRS 505: Vadose Zone Hydrology

Lecture 9

9/19/2023

Today: Richards' Equation
Reading: Chapter 11 (Pinder & Celia, 2006).

Review of Lecture 8

- Derivation of governing equations for two-phase flow in porous media
  - Mass conservation for phase α ($\alpha = w$ or nw)

$$\frac{\partial}{\partial t}(\rho_\alpha \phi s_\alpha) + \nabla \cdot (\rho_\alpha \mathbf{q}_\alpha) = 0$$

  - Flux law: Extended Darcy's law for two-phase flow in porous media

$$\mathbf{q}_\alpha = -\frac{k_{r,\alpha}\mathbf{k}}{\mu_\alpha}\nabla(p_\alpha + \rho_\alpha g z)$$

Recall Darcy's law for single-phase (saturated flow): $\mathbf{q} = -\frac{\mathbf{k}}{\mu_w}\nabla(p_w + \rho_w g z)$

- Richards' assumptions
  - $\mu_a \ll \mu_w$ and $\rho_a \ll \rho_w$. $\rho_a \ll \rho_w$ helps to simply, but it is NOT required. Why?
  - Reduces the general two-phase flow equation to only one equation for water flow (i.e., Richards' Equation)

Richards' Equation

Convert water pressure to water pressure head as $\psi_w = p_w/(\rho_w g)$

$$\phi \frac{\partial S_w}{\partial t} - \nabla \cdot \left[k_{rw} \frac{\mathbf{k}\rho_w g}{\mu_w} (\nabla \psi_w + \nabla \mathrm{z}) \right] = 0$$

$\mathbf{K}_{sat}^{w} \equiv \frac{\mathbf{k}\rho_w g}{\mu_w}$, then

$$\phi \frac{\partial S_w}{\partial t} - \nabla \cdot [k_{rw} \mathbf{K}_{sat}^{w} (\nabla \psi_w + \nabla \mathrm{z})] = 0$$

$\theta_w = \phi S_w$, then

$$\frac{\partial \theta_w}{\partial t} - \nabla \cdot [\mathbf{K}_{sat}^{w} k_{rw} (\nabla \psi_w + \nabla \mathrm{z})] = 0$$

3D Richards' Equation written in a "mixed" form, i.e., both moisture content and pressure head appear explicitly in the equation.

Richards' Equation

$$\frac{\partial\theta_w}{\partial t} = \frac{\mathrm{d}\theta_w}{\mathrm{d}\psi_w}\frac{\partial\psi_w}{\partial t} = C(\psi_w)\frac{\partial\psi_w}{\partial t}, \quad C(\psi_w) \equiv \frac{\mathrm{d}\theta_w}{\mathrm{d}\psi_w}$$

Specific moisture capacity

$$\Rightarrow C(\psi_w)\frac{\partial\psi_w}{\partial t} - \nabla\cdot[k_{rw}\mathbf{K}_{sat}^{w}(\nabla\psi_w + \nabla \mathrm{z})] = 0$$

"ψ-based" or "pressure-based" form

$$\nabla\psi_w = \frac{\mathrm{d}\psi_w}{\mathrm{d}\theta_w}\nabla\theta_w, \quad \boldsymbol{D}(\theta_w) \equiv \frac{k_{rw}\mathbf{K}_{sat}^{w}}{\mathrm{d}\theta_w/\mathrm{d}\psi_w}$$

Soil moisture diffusivity

$$\Rightarrow \frac{\partial\theta_w}{\partial t} - \nabla\cdot\left[k_{rw}\mathbf{K}_{sat}^{w}\left(\frac{\mathrm{d}\psi_w}{\mathrm{d}\theta_w}\nabla\theta_w + \nabla \mathrm{z}\right)\right] = 0$$

$$\Rightarrow \frac{\partial\theta_w}{\partial t} - \nabla\cdot\left[\frac{k_{rw}\mathbf{K}_{sat}^{w}}{\mathrm{d}\theta_w/\mathrm{d}\psi_w}\nabla\theta_w + k_{rw}\mathbf{K}_{sat}^{w}\nabla \mathrm{z}\right] = 0$$

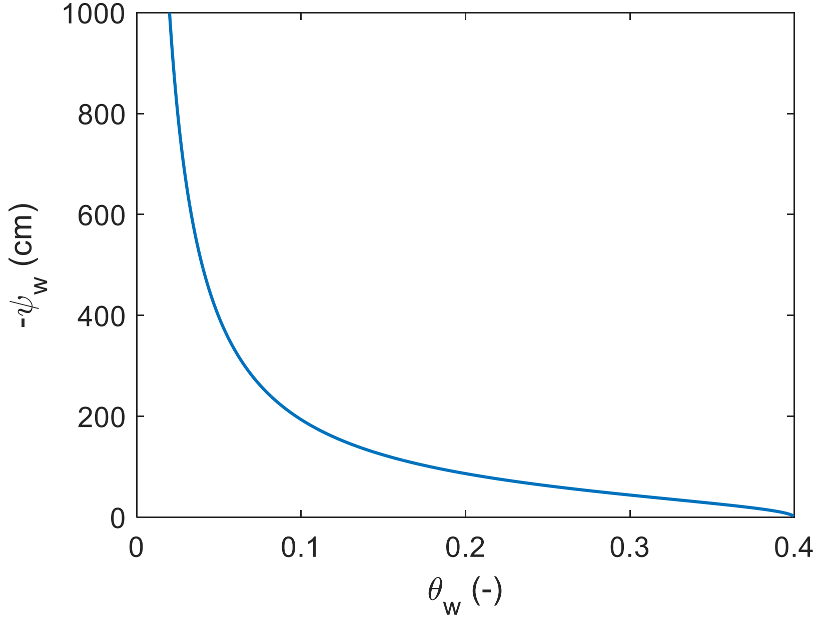
$$\Rightarrow \frac{\partial\theta_w}{\partial t} - \nabla\cdot[\boldsymbol{D}(\theta_w)\nabla\theta_w] - \nabla\cdot[k_{rw}\mathbf{K}_{sat}^{w}\nabla \mathrm{z}] = 0$$

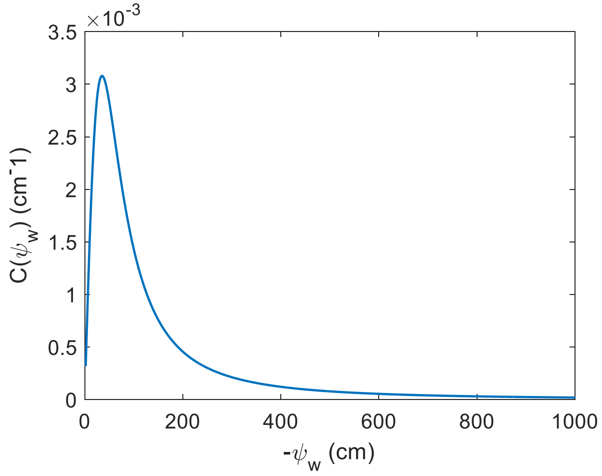
"θ-based" or "moisture content-based" form

This term represents moisture movement driven by capillary force, which acts like a diffusive process, i.e., water flux is driven by the gradient of moisture where $\boldsymbol{D}(\theta_w)$ is the (nonlinear) diffusion coefficient.

$C(\psi_w)$ and $\boldsymbol{D}(\theta_w)$

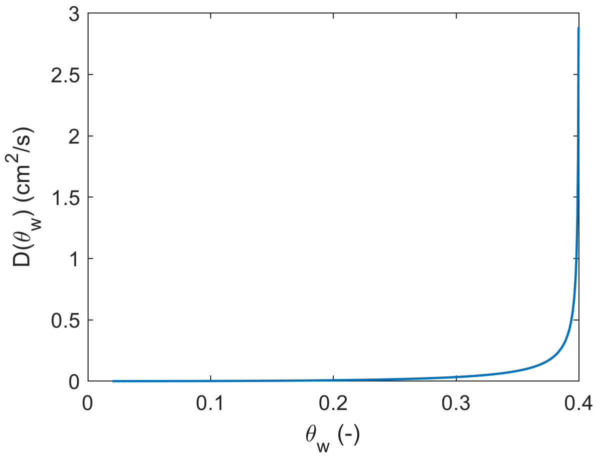
$$C(\psi_w) \equiv \frac{d\theta_w}{d\psi_w}$$ ← Specific moisture capacity





$$\boldsymbol{D}(\theta_w) \equiv \frac{\mathbf{K}^w_{sat} k_{rw}}{\mathrm{d}\theta_w/\mathrm{d}\psi_w}$$ ← Soil moisture diffusivity

$C(\psi_w)$



Some Remarks on Richards' Equation

- θ-based form is appropriate for <u>unsaturated</u> systems, but not for systems that include <u>saturated</u> zones.

  (1) $\boldsymbol{D}(\theta_w) \equiv \frac{k_{rw}\mathbf{K}_{sat}^{w}}{\mathrm{d}\theta_w/\mathrm{d}\psi_w}$ becomes unbounded as $\frac{\mathrm{d}\theta_w}{\mathrm{d}\psi_w} \to 0$ under saturated condition.

  (2) In saturated <u>heterogeneous</u> porous media, $\nabla\theta_w$ does not represent a driving force.

- When <u>saturated</u> conditions occur, the mixed form or the pressure head-based form should be used.

- Soil and fluid compressibility are neglected. They can be included to Richards' equation by relaxing these assumptions.

$$\boldsymbol{\nabla}\cdot(\rho_w \boldsymbol{q}_w) = \rho_w \boldsymbol{\nabla}\cdot\boldsymbol{q}_w + \cancelto{0}{\boldsymbol{q}_w \nabla\rho_w} \approx \rho_w \boldsymbol{\nabla}\cdot\boldsymbol{q}_w$$

$$\frac{\partial}{\partial t}(\rho_w \phi s_w) + \boldsymbol{\nabla}\cdot(\rho_w \boldsymbol{q}_w) = 0.$$

$$\boldsymbol{q}_w \nabla\rho_w \ll \rho_w \boldsymbol{\nabla}\cdot\boldsymbol{q}_w$$

Water is only slightly compressible.

$$\frac{\partial}{\partial t}(\rho_w \phi s_w) = \rho_w \phi \frac{\partial s_w}{\partial t} + \rho_w s_w \frac{\partial \phi}{\partial t} + \phi s_w \frac{\partial \rho_w}{\partial t} = \rho_w \phi \frac{\partial s_w}{\partial t} + (\rho_w s_w \frac{\partial \phi}{\partial p_w} + \phi s_w \frac{\partial \rho_w}{\partial p_w})\frac{\partial p_w}{\partial t}$$

$$\frac{\partial \phi}{\partial p_w}\frac{\partial p_w}{\partial t} \qquad \frac{\partial \rho_w}{\partial p_w}\frac{\partial p_w}{\partial t}$$

<u>Note</u>: Contribution to the water storage term by compressibility becomes more important under saturated conditions.

Examples of Air-Water Flow (1D)

BC at the bottom: fixed pressures for air and water

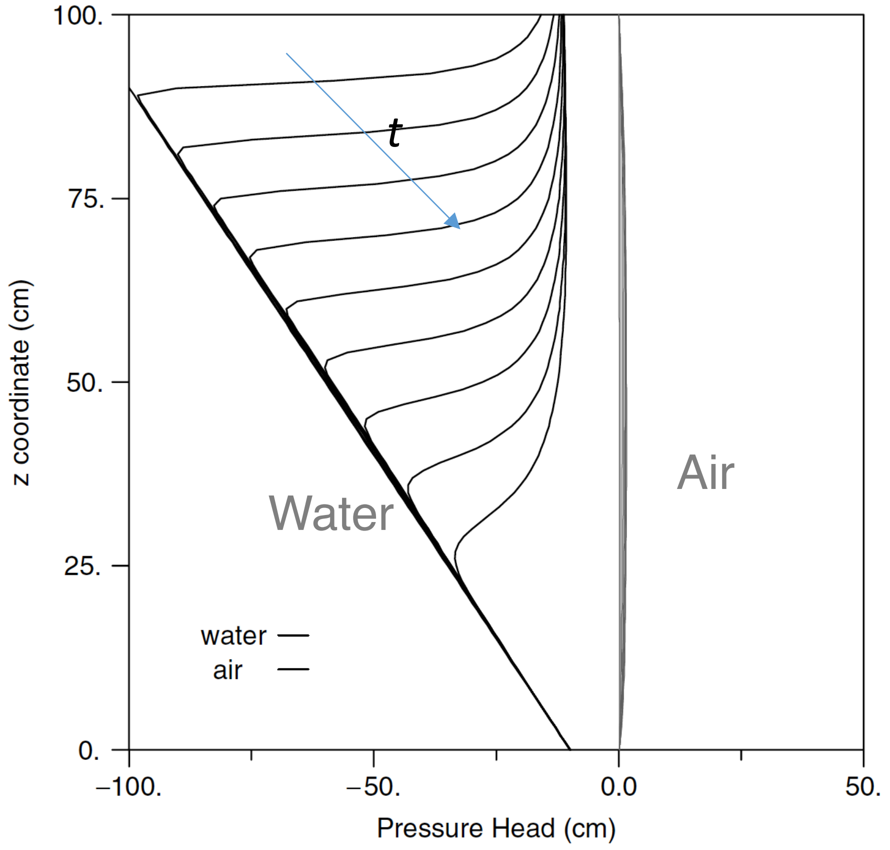


FIGURE 11.6. Plot of both water and air pressure heads as a function of depth, for different values of time. The water pressure head changes over a range of about 100 cm of water, while the air pressure head changes by only a small fraction of that amount (from [15]).

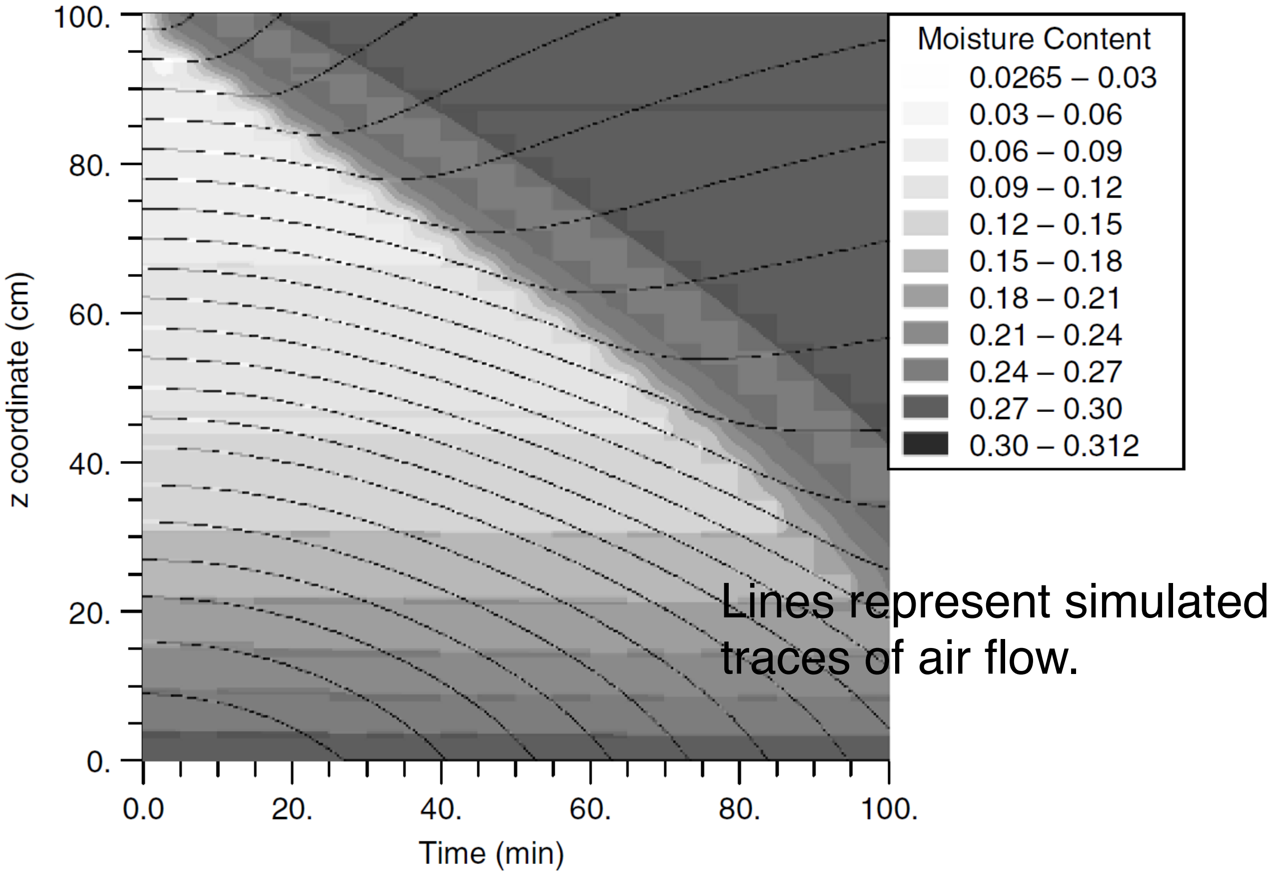


FIGURE 11.7. Water infiltration and air movement (from [15]).

Pinder & Celia (2006)

- Air pressure buildup due to water infiltration is very little, but there is clear dynamic air flow

Examples of Air-Water Flow (1D)

BC at the bottom: impervious for fluids

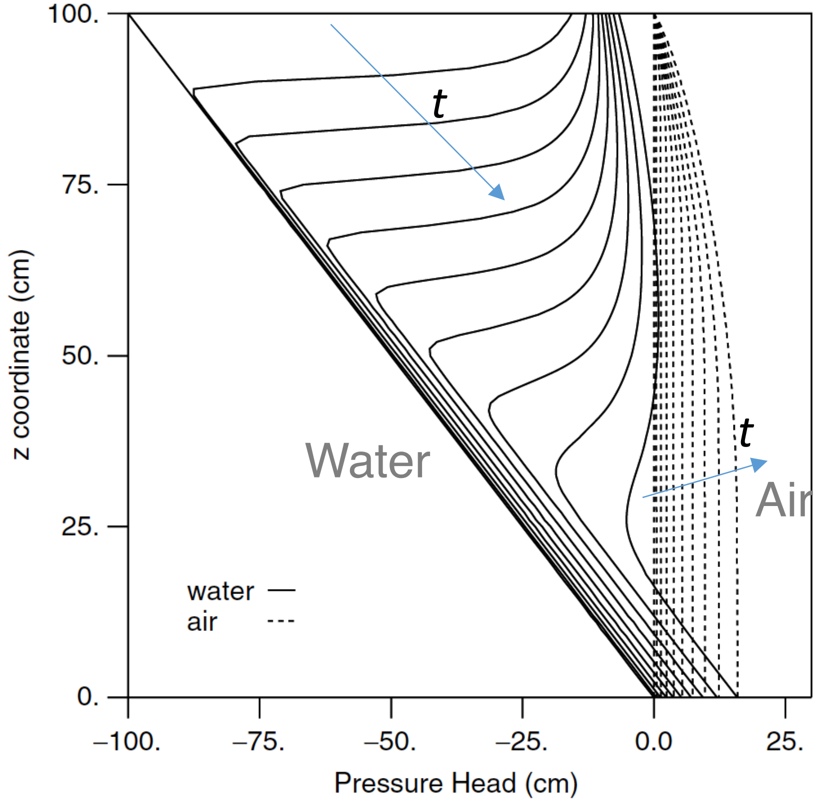


FIGURE 11.8. Plot of water and air pressure heads as a function of depth, for different times, for a system with impervious bottom boundary (from [15]).

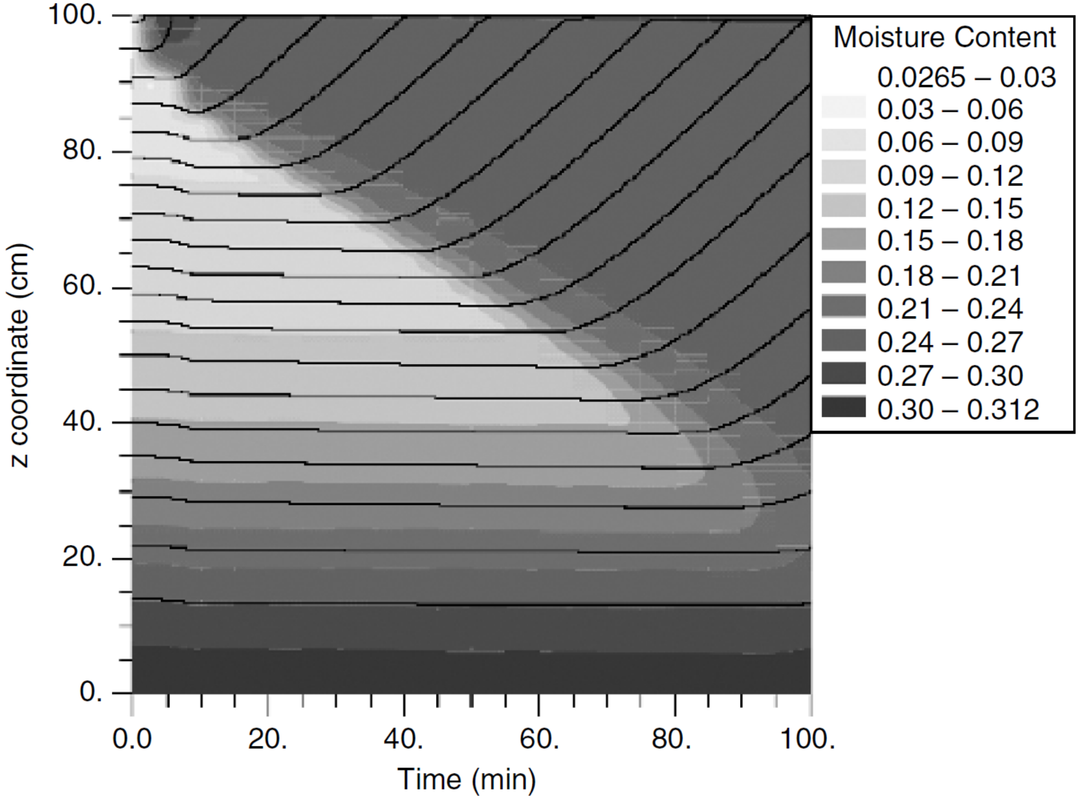
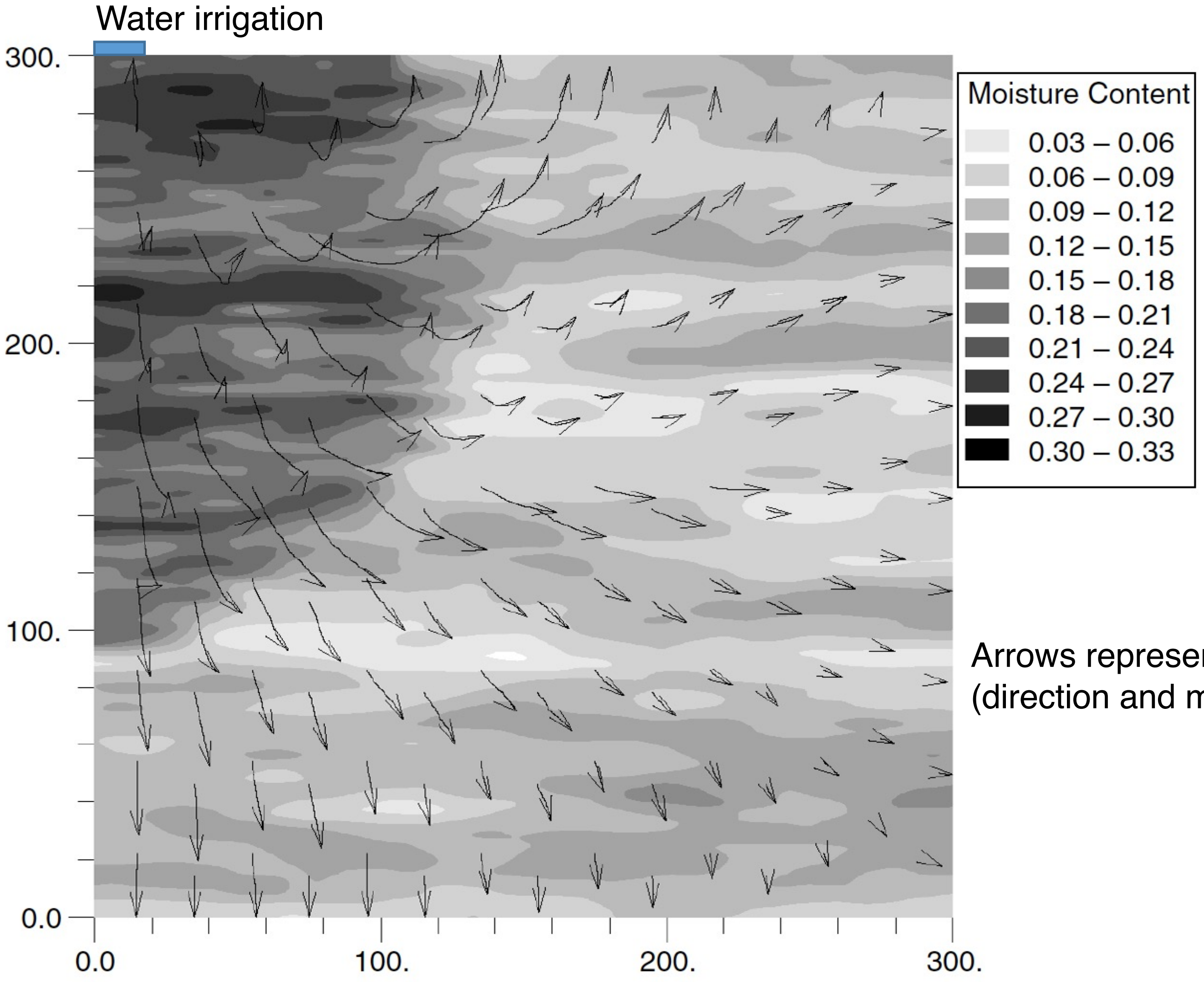


FIGURE 11.9. Plot showing water saturation as a function of depth and time (from [15]).

- Notable air pressure buildup due to water infiltration. There is also clear dynamic air flow.
- Richards' equation would be invalid.

Pinder & Celia (2006)

Examples of Air-Water Flow (2D)



Arrows represent air flow (direction and magnitude)

FIGURE 11.10. Plot of moisture content (shaded scale) at a given time, with air pathways superimposed, for the case of infiltration into a heterogeneous two-dimensional system and air pathway traces (from [17]).

Pinder & Celia (2006)

1D Richards' Equation

The three forms of Richards' Equation in the vertical dimension. $K \equiv K_{sat}^{w} k_{rw}$

$$\frac{\partial \theta_w}{\partial t} - \frac{\partial}{\partial z}\left(K \frac{\partial \psi_w}{\partial z}\right) - \frac{\partial K}{\partial z} = 0$$

"Mixed" form

$$C(\psi_w)\frac{\partial \psi_w}{\partial t} - \frac{\partial}{\partial z}\left(K \frac{\partial \psi_w}{\partial z}\right) - \frac{\partial K}{\partial z} = 0$$

"ψ-based" or "pressure-based" form

$$\frac{\partial \theta_w}{\partial t} - \frac{\partial}{\partial z}\left(D \frac{\partial \theta_w}{\partial z}\right) - \frac{\partial K}{\partial z} = 0$$

"θ-based" or "moisture content-based" form

The three forms of Richards' Equation in the horizontal dimension.

$$\frac{\partial \theta_w}{\partial t} - \frac{\partial}{\partial x}\left(K \frac{\partial \psi_w}{\partial x}\right) = 0$$

"Mixed" form

$$C(\psi_w)\frac{\partial \psi_w}{\partial t} - \frac{\partial}{\partial x}\left(K \frac{\partial \psi_w}{\partial x}\right) = 0$$

"ψ-based" or "pressure-based" form

$$\frac{\partial \theta_w}{\partial t} - \frac{\partial}{\partial x}\left(D \frac{\partial \theta_w}{\partial x}\right) = 0$$

"θ-based" or "moisture content-based" form

Steady-State 1D Richards' Equation

In the vertical dimension.

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)+\frac{\mathrm{d}K}{\mathrm{d}z}=0$$

"ψ-based" or "pressure-based" form

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(D\frac{\mathrm{d}\theta_w}{\mathrm{d}z}\right)+\frac{\mathrm{d}K}{\mathrm{d}z}=0$$

"θ-based" or "moisture content-based" form

In the horizontal dimension.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}x}\right)=0$$

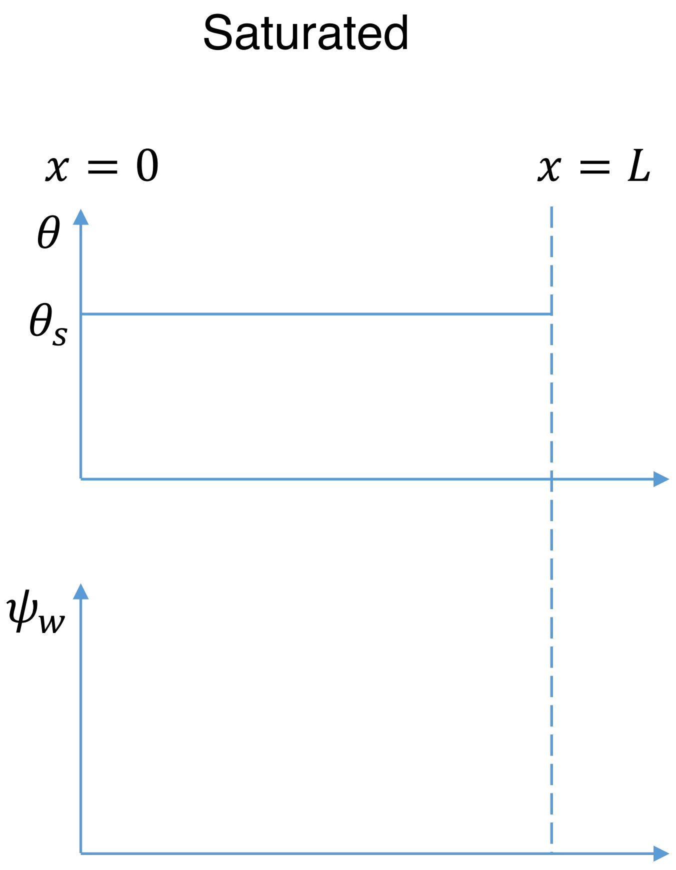
"ψ-based" or "pressure-based" form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(D\frac{\mathrm{d}\theta_w}{\mathrm{d}x}\right)=0$$

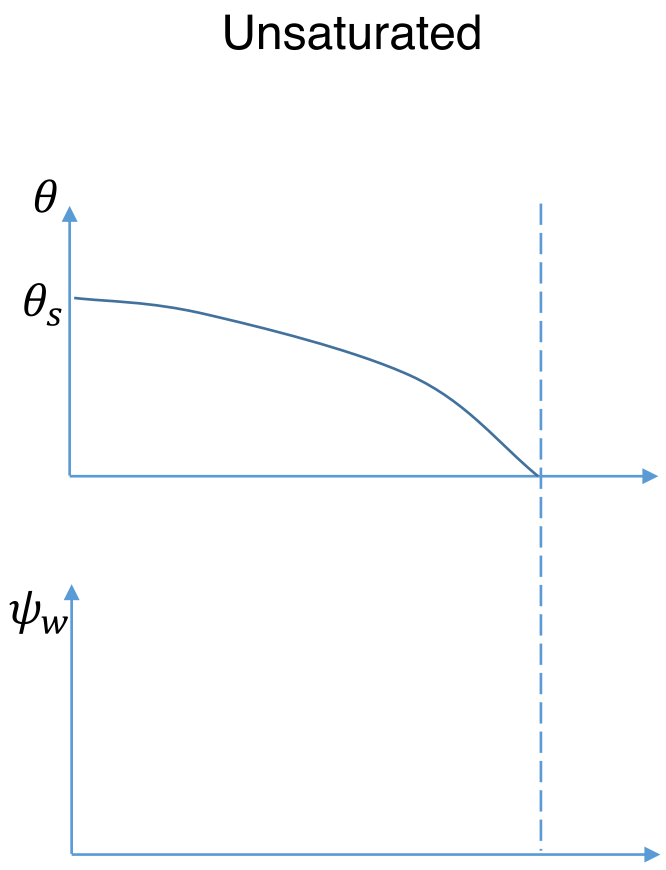
"θ-based" or "moisture content-based" form

Steady-State Unsaturated Flow

Horizontal flow (Homogeneous column)



$\psi_w = 0$ at $x = L$



$\psi_w = 0$ at $x = 0$