HWRS 561b Homework #4

- Assigned: Monday, 23 March 2026
- Due: Thursday, 2 April 2026 (upload answers in PDF or Jupyter Notebook to D2L)
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- Semester: Spring 2026

1. (20 points) Richards assumptions.

   Consider two infiltration experiments in a 1D soil column:

   - Case A: the bottom boundary has fixed air and water pressures.
   - Case B: the bottom boundary is impervious to both fluids.
   - (a) (6 points) For each case, state whether Richards equation is expected to be valid, and justify your answer based on the underlying assumptions of Richards equation.
   - (b) (7 points) In Case A, the simulated air velocity is nonzero. Explain why this does not necessarily violate the assumptions of Richards equation.
   - (c) (7 points) Give one realistic field situation in which the assumptions of Richards equation may become weak or invalid. In that case, the air pressure cannot be assumed zero. How would you still solve for the evolution of water saturation?

2. (20 points) Steady-state 1D unsaturated flow in a horizontal column.

   Consider steady flow in a homogeneous horizontal column of length $L = 1.0$ m. Let x increase from inlet to outlet. The pressure heads are $h(0) = 0$ m and $h(L) = -0.8$ m. Assume

   $$K(h) = K_{sat}\, e^{\alpha h}, \qquad K_{sat} = 2.0 \times 10^{-5} \text{ m/s}, \quad \alpha = 3.0 \text{ m}^{-1}.$$

   - (a) (4 points) Write the governing equation for this 1D problem in terms of $h(x)$.
   - (b) (6 points) Using the boundary conditions, derive and numerically compute the steady Darcy flux q. Hint: 1) separate h and x onto opposite sides of the governing equation and use definite integration; 2) $(\frac{1}{\alpha}e^{\alpha h})' = e^{\alpha h}$.
   - (c) (4 points) Sketch the expected shape of $h(x)$ and explain why it is nonlinear, in contrast to saturated flow.
   - (d) (6 points) Once q is known, solve for $h(x)$ analytically or, if you do not know how to solve it analytically (which is fine), explain how $h(x)$ may be solved. In this case, you do not need to carry out the full solution; just describe the procedure clearly.

3. (20 points) Steady-state 1D vertical flow in layered media.

   A vertical column has two layers of equal thickness under steady downward flow:

   - Layer 1 (top): coarse sand
   - Layer 2 (bottom): fine sand

   Consider a second configuration in which the order is reversed (fine over coarse).

- (a) (8 points) For both configurations, discuss whether hydraulic head H, pressure head h, and water saturation S_w are continuous or discontinuous at the interface. Also sketch the qualitative profile of S_w across the interface.
- (b) (12 points) Now consider a different scenario. Suppose that in both configurations, both layers are initially at residual water saturation. Under these conditions, the capillary pressure in the finer layer is greater than that in the coarse layer. A moderate infiltration flux (smaller than saturated conductivity for both layers) is then imposed at the top boundary. Describe qualitatively the expected behavior at the interface for each configuration.

4. (20 points) Cell-centered finite difference solution of the steady-state 1D Richards equation (4 boxes).

   Let z be positive upward. Consider the steady-state pressure head-based 1D Richards equation

$$\frac{d}{dz}\left[K(h)\frac{dh}{dz}\right] + \frac{dK(h)}{dz} = 0.$$

   on the domain $z \in [0, 1.0]$ m with boundary conditions $h(0) = 0$ m and $h(1.0) = -1.0$ m.

   Use the following sand hydraulic properties with the van Genuchten-Mualem model:

$$\theta_r = 0.045, \qquad \theta_s = 0.43, \qquad \alpha = 0.145 \text{ cm}^{-1}, \qquad n = 2.68, \qquad m = 1 - \frac{1}{n} = 0.627, \qquad l = 0.5,$$

$$K_s = 712.8 \text{ cm/day} = 29.7 \text{ cm/h} = 8.25 \times 10^{-5} \text{ m/s}.$$

   For the nonlinear system in part (b), use

$$\theta(h) = \theta_r + \frac{\theta_s - \theta_r}{(1 + |\alpha h|^n)^m}, \qquad h < 0,$$

   with $\theta = \theta_s$ for $h \geq 0$, and

$$K(S_e) = K_s S_e^l \left[1 - \left(1 - S_e^{1/m}\right)^m\right]^2, \qquad S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r}.$$

   Discretize the domain into 4 uniform boxes ($\Delta z = 0.25$ m) using a cell-centered finite difference scheme, with unknown pressure heads h_1, h_2, h_3, h_4 at the cell centers. For this problem, assume $h_1 = h(0.125) = -0.125$ m and $h_1 = h(0.875) = -0.875$ m to represent the boundary condition, as an approximation.

   Evaluate the conductivity at each interior face by arithmetic averaging of the conductivities at the two adjacent cell centers. At the boundary faces, evaluate conductivity directly from the boundary heads $h = 0$ m and $h = -1.0$ m.

   For the frozen-conductivity calculation in part (c), assume a linear initial head profile so that

$$h_1 = -0.125 \text{ m}, \qquad h_2 = -0.375 \text{ m}, \qquad h_3 = -0.625 \text{ m}, \qquad h_4 = -0.875 \text{ m}.$$

This gives the following frozen face conductivities:

$$K_{1/2} = 8.25\times10^{-5},\ K_{3/2} = 2.83\times10^{-7},\ K_{5/2} = 4.57\times10^{-10},\ K_{7/2} = 2.11\times10^{-11},\ K_{9/2} = 2.04\times10^{-12}\ \text{m/s}.$$

Here, faces $1/2$ and $9/2$ are boundary faces, and faces $3/2, 5/2, 7/2$ are interior faces.

No coding is required for this problem. Derive and solve the system by hand.

- (a) (8 points) Using cell-centered finite differences, derive the algebraic equation for each cell ($i = 1, 2, 3, 4$), including the gravity term written as $\left(K_{i+1/2} - K_{i-1/2}\right)/\Delta z$.
- (b) (6 points) Write the coupled nonlinear system in residual form, $\mathbf{F}(\mathbf{h}) = \mathbf{0}$, for $\mathbf{h} = (h_1, h_2, h_3, h_4)^T$ after substituting the boundary heads h_1 and h_4 at $z = 0.125$ and $z = 0.875$ m.
- (c) (6 points) Using the frozen face-conductivity values above, assemble the resulting 4×4 linear system and solve it by hand. Report h_1, h_2, h_3, and h_4. Then compare the computed heads with the assumed linear initial head profile used to freeze the conductivities. If they differ, explain why the one-step solution is only approximate and describe how you would improve it. No further calculation is required.

5. (20 points) Transient drainage in a 100 cm vertical sand column.

   Consider a 100 cm long vertical soil column filled with sand. Use the same van Genuchten-Mualem hydraulic parameters from problem 4.

   Initially, the entire column is at a uniform pressure head of $h = 0$ cm. At the top boundary, impose zero water flux. At the bottom boundary, impose free drainage.

   - (a) (8 points) Using a hand-calculation approach, estimate the drainage progression over 24 hours with a time step of 1 hour. Assume that within each 1-hour step, the flux is constant and that the hydraulic conductivity K is determined from the water content at the beginning of that time step. Also assume that water saturation remains uniform throughout the column. You may use a spreadsheet or write a short Python script to carry out the calculations.
   - (b) (6 points) Repeat the calculation in part (a) using a time step of 0.1 hour. Compare the two sets of results and explain any significant differences.
   - (c) (6 points) Suppose that the calculation using a time step of 0.1 hour is effectively converged, meaning that using an even smaller time step would not noticeably change the result. Based on that converged solution: 1) describe how drainage changes over time and explain why; 2) between a sandy soil and a clayey soil, discuss for which soil the assumption of uniform water saturation during drainage is likely to be more reasonable.