HWRS 561b
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HWRS 505: Vadose Zone Hydrology

Steady-state 1D unsat flow (Part 1)

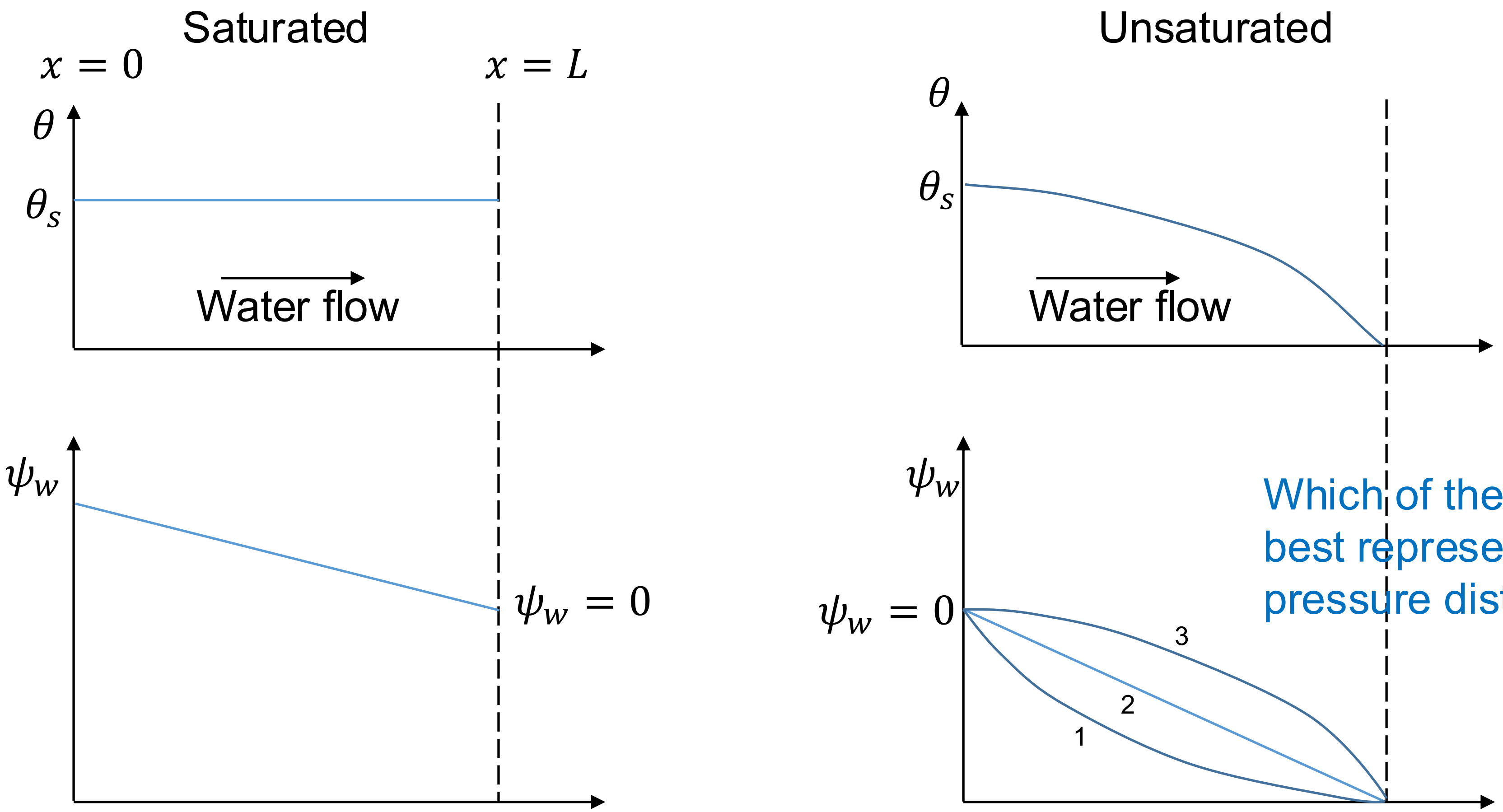
Agenda: Steady-state unsaturated flow
Reading: Chapter 11 (Pinder & Celia, 2006)

Review

- Three forms of Richards’ equation
  - Mixed form
  - Pressure head-based form
    - ✓ Specific moisture capacity
  - Water content-based form
    - ✓ Soil moisture diffusivity (What does the equation have to do with “diffusion”?)
    - ✓ Cannot be used if the domain involves saturated water flow
  - How to include soil and fluid compressibility?
- Richards’ assumptions
  - Air pressure remains almost zero everywhere, but air does move.
  - Does “air movement” make the Richards’ equation invalid? No, as long as air pressure remains almost zero everywhere.

Steady-State Unsaturated Flow

Horizontal flow (Homogeneous column)



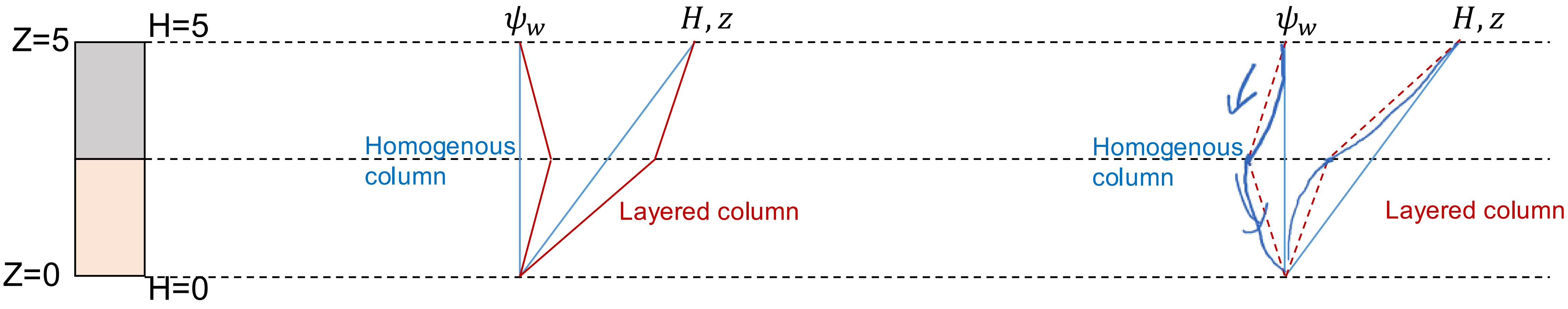
Which of the curves (1, 2, or 3) best represents the water pressure distribution?

Answer: 3

- ➢ Unsaturated flow involves nonlinearities that make their behaviors differ from that of the saturated flow

Steady-State Unsaturated Flow

Vertical flow (in layered columns)



Coarse over fine

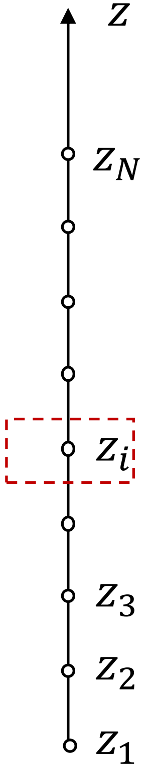
Fine over coarse

- Hydraulic head and pressure head are both continuous in space.
- Is water saturation continuous?

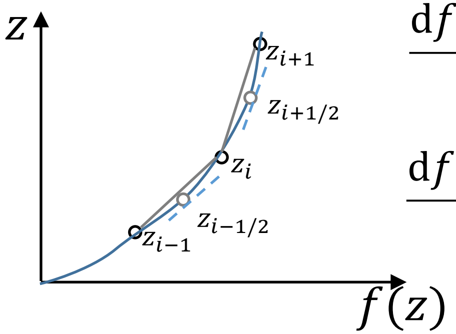
Steady-State Unsaturated Flow: Numerical Soln.

$$\cancelto{0}{\frac{\partial\theta_w}{\partial t}} - \frac{\partial}{\partial z}\left(K\frac{\partial\psi_w}{\partial z}\right) - \frac{\partial K}{\partial z} = 0 \xrightarrow{\text{Steady-state}} \overset{①}{\frac{\mathrm{d}}{\mathrm{dz}}\left(K\frac{\mathrm{d}\psi_w}{\mathrm{dz}}\right)} + \overset{②}{\frac{\mathrm{d}K}{\mathrm{dz}}} = 0 \qquad (1)$$

- Equation (1) is a second-order ordinary differential equation in 1D.
- It is nonlinear because $K = K(\psi_w)$ is a nonlinear function.
- To solve it, we need two boundary conditions and we need to do it in an iterative procedure.



- How to solve this 1D **nonlinear ordinary differential equation**?
- Key idea: Divide the domain into many boxes and convert the **differential equation** to a system of nonlinear **algebraic equations**.
- Technique: Use finite difference to approximate derivatives

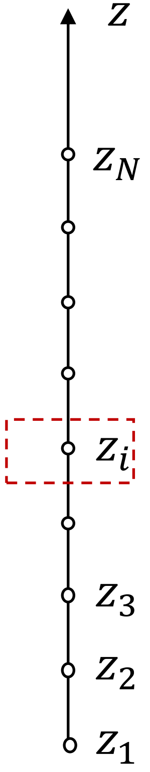


$$\frac{\mathrm{d}f(z_{i+1/2})}{\mathrm{dz}} \approx \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}$$

$$\frac{\mathrm{d}f(z_{i-1/2})}{\mathrm{dz}} \approx \frac{f(z_i) - f(z_{i-1})}{z_i - z_{i-1}}$$

Steady-State Unsaturated Flow: Numerical Soln.

$$\cancelto{0}{\frac{\partial \theta_w}{\partial t}} - \frac{\partial}{\partial z}\left(K\frac{\partial \psi_w}{\partial z}\right) - \frac{\partial K}{\partial z} = 0 \xrightarrow{\text{Steady-state}} \overset{\textcircled{1}}{\frac{\mathrm{d}}{\mathrm{d}z}\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)} + \overset{\textcircled{2}}{\frac{\mathrm{d}K}{\mathrm{d}z}} = 0 \qquad (1)$$



$$\overset{\textcircled{1}}{\frac{\mathrm{d}}{\mathrm{d}z}\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)}\Bigg|_{z_i} \approx \frac{\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)_{i+1/2} - \left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)_{i-1/2}}{\Delta z} \approx \frac{K_{i+1/2}\frac{\psi_{w,i+1}-\psi_{w,i}}{\Delta z} - K_{i-1/2}\frac{\psi_{w,i}-\psi_{w,i-1}}{\Delta z}}{\Delta z}$$

$$= \frac{K_{i+1/2}(\psi_{w,i+1}-\psi_{w,i}) - K_{i-1/2}(\psi_{w,i}-\psi_{w,i-1})}{\Delta z^2}$$

$$\overset{\textcircled{2}}{\frac{\mathrm{d}K}{\mathrm{d}z}}\Bigg|_{z_i} \approx \frac{K_{i+1/2} - K_{i-1/2}}{\Delta z}$$

$$\Rightarrow \frac{K_{i+1/2}}{\Delta z^2}(\psi_{w,i+1}-\psi_{w,i}) - \frac{K_{i-1/2}}{\Delta z^2}(\psi_{w,i}-\psi_{w,i-1}) + K_{i+1/2}\frac{1}{\Delta z} - K_{i-1/2}\frac{1}{\Delta z} = 0$$

$$\Rightarrow \frac{K_{i+1/2}}{\Delta z^2}\psi_{w,i+1} - \left(\frac{K_{i+1/2}}{\Delta z^2} + \frac{K_{i-1/2}}{\Delta z^2}\right)\psi_{w,i} + \frac{K_{i-1/2}}{\Delta z^2}\psi_{w,i-1} + K_{i+1/2}\frac{1}{\Delta z} - K_{i-1/2}\frac{1}{\Delta z} = 0$$

This is an algebraic equation with 3 unknowns. We can write such an algebraic equation for each node or box and we can get N algebraic equations for the N unknowns $(\psi_{w,1}, \psi_{w,2}, \ldots, \psi_{w,N})$

Steady-State Unsaturated Flow: Numerical Soln.

In matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & & & & \\ \frac{K_{3/2}}{\Delta z^2} & -\frac{\left(K_{3/2}+K_{5/2}\right)}{\Delta z^2} & \frac{K_{5/2}}{\Delta z^2} & & & & \\ & \frac{K_{5/2}}{\Delta z^2} & -\frac{\left(K_{5/2}+K_{7/2}\right)}{\Delta z^2} & \frac{K_{7/2}}{\Delta z^2} & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \frac{K_{i-1/2}}{\Delta z^2} & -\frac{\left(K_{i-1/2}+K_{i+1/2}\right)}{\Delta z^2} & \frac{K_{i+1/2}}{\Delta z^2} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \frac{K_{n-1-1/2}}{\Delta z^2} & -\frac{\left(K_{n-1-1/2}+K_{n-1+1/2}\right)}{\Delta z^2} & \frac{K_{n-1+1/2}}{\Delta z^2} \\ & & & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_{w,1} \\ \psi_{w,2} \\ \psi_{w,3} \\ \vdots \\ \psi_{w,i} \\ \vdots \\ \psi_{w,n-1} \\ \psi_{w,n} \end{pmatrix} \begin{pmatrix} B.C. \\ \frac{K_{3/2}-K_{5/2}}{\Delta z} \\ \frac{K_{5/2}-K_{7/2}}{\Delta z} \\ \vdots \\ \frac{K_{i-1/2}-K_{i+1/2}}{\Delta z} \\ \vdots \\ \frac{K_{n-1-1/2}-K_{n-1+1/2}}{\Delta z} \\ B.C. \end{pmatrix}$$

$\Rightarrow \boldsymbol{F}(\boldsymbol{\psi}_w) \cdot \boldsymbol{\psi}_w = \boldsymbol{R}$ How to solve this nonlinear system of algebraic equations?

=> Employ an iterative methods

Steady-State Unsaturated Flow: Numerical Soln.

$$\frac{\partial \theta_w}{\partial t} + \frac{\partial}{\partial z}\left(-K(\frac{\partial \psi_w}{\partial z} + 1)\right) = 0 \qquad q_z = -K(\frac{\partial \psi_w}{\partial z} + 1)$$

Assumes z positive upward

$$\frac{\partial \theta_w}{\partial t} + \frac{\partial}{\partial z}\left(-K(\frac{\partial \psi_w}{\partial z} - 1)\right) = 0 \qquad q_z = -K(\frac{\partial \psi_w}{\partial z} - 1)$$

Assumes z positive downward

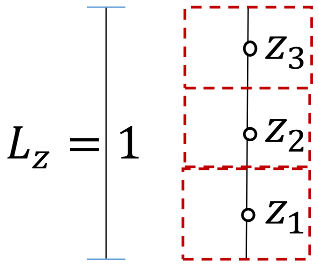
Which equation is correct?

Now, we assume z is positive upward.
Which equation is correct?

Steady-State Unsaturated Flow: Numerical Soln.

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)+\frac{\mathrm{d}K}{\mathrm{d}z}=0$$

$$\psi_w(z_3) = 2$$



$$\psi_w(z_1) = 0$$

$$\Delta z = z_2 - z_1 = z_3 - z_2 = 1/3$$

Equation for box ①:

$$\psi_{w,1} = 0$$

Equation for box ③:

$$\psi_{w,3} = 2$$

Equation for box ②:

$$\frac{\left(K\frac{d\psi_w}{dz}\right)\Big|_{z+1/2} - \left(K\frac{d\psi_w}{dz}\right)_{z-1/2}}{\Delta z} + \frac{K_{z+1/2}-K_{z-1/2}}{\Delta z} = 0$$

$$\Rightarrow \frac{K_{z+1/2}\frac{\psi_{w,3}-\psi_{w,2}}{\Delta z} - K_{z-1/2}\frac{\psi_{w,2}-\psi_{w,1}}{\Delta z}}{\Delta z} + \frac{K_{z+1/2}-K_{z-1/2}}{\Delta z} = 0$$

$$\Rightarrow \frac{K_{z+1/2}}{\Delta z^2}\psi_{w,3} + \left(-\frac{K_{z+1/2}}{\Delta z^2} + \frac{K_{z-1/2}}{\Delta z^2}\right)\psi_{w,2} + \frac{K_{z-1/2}}{\Delta z^2}\psi_{w,1} = -\frac{K_{z+1/2}-K_{z-1/2}}{\Delta z}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{K_{z-1/2}}{\Delta z^2} & -\frac{K_{z+1/2}}{\Delta z^2}+\frac{K_{z-1/2}}{\Delta z^2} & \frac{K_{z+1/2}}{\Delta z^2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \psi_{w,1} \\ \psi_{w,2} \\ \psi_{w,3} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{K_{z+1/2}-K_{z-1/2}}{\Delta z} \\ 2 \end{bmatrix}$$

For our given B.C.:

$$\psi_{w,2} = 1$$

⇒ Everywhere is saturated

For another B.C.:

e.g.: $\psi_{w,1} = -3$

$\psi_{w,3} = -1$

Need to solve iteratively