

HWRS 505: Vadose Zone Hydrology

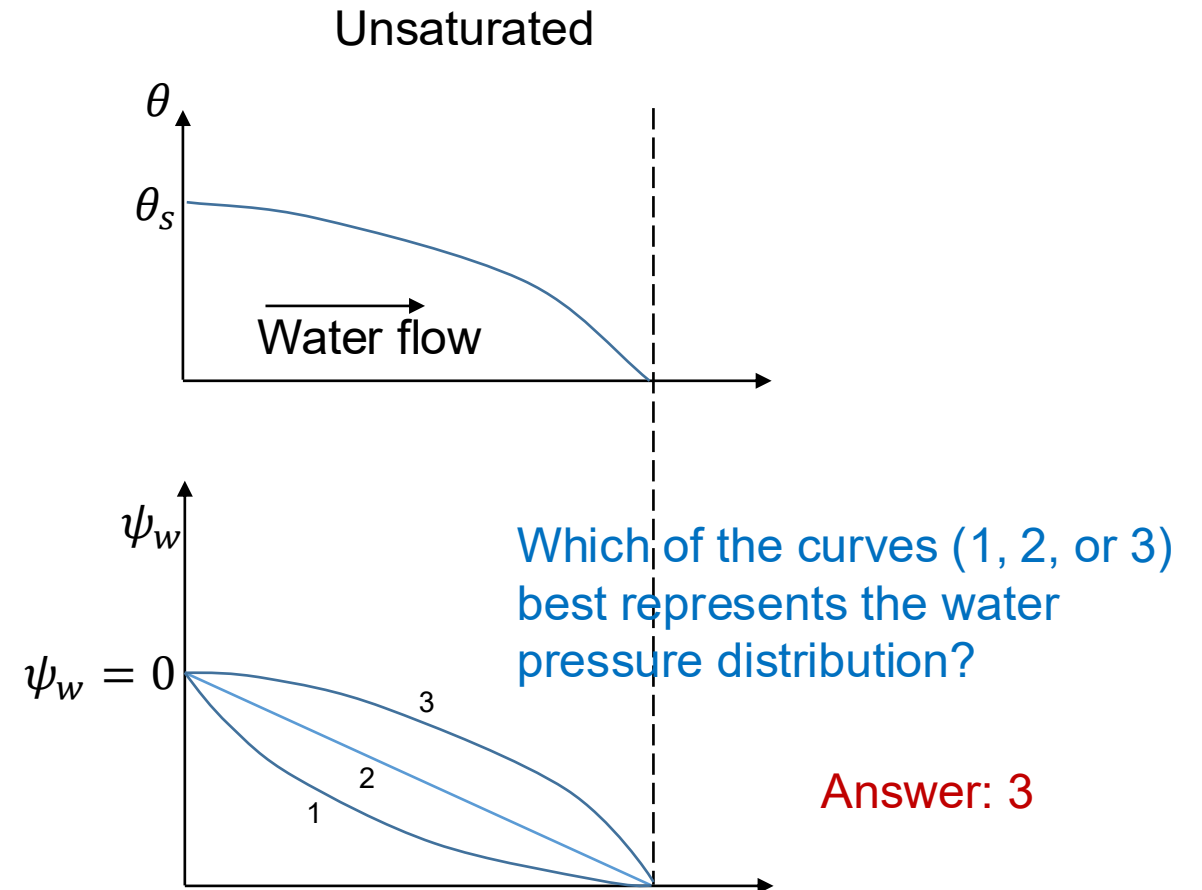
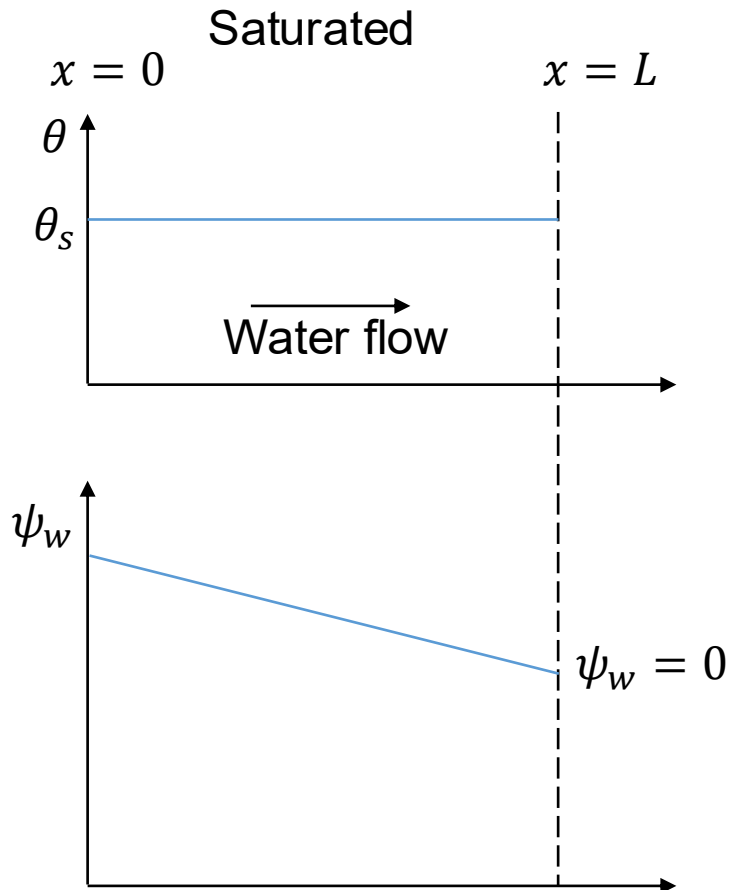
Steady-state 1D unsat flow (Part 1)

Agenda: Steady-state unsaturated flow
Reading: Chapter 11 (Pinder & Celia, 2006)

- ❖ Three forms of Richards' equation
 - Mixed form
 - Pressure head-based form
 - ✓ Specific moisture capacity
 - Water content-based form
 - ✓ Soil moisture diffusivity (What does the equation have to do with “diffusion”?)
 - ✓ Cannot be used if the domain involves saturated water flow
 - How to include soil and fluid compressibility?
- ❖ Richards' assumptions
 - Air pressure remains almost zero everywhere, but air does move.
 - Does “air movement” make the Richards' equation invalid? No, as long as air pressure remains almost zero everywhere.

Steady-State Unsaturated Flow

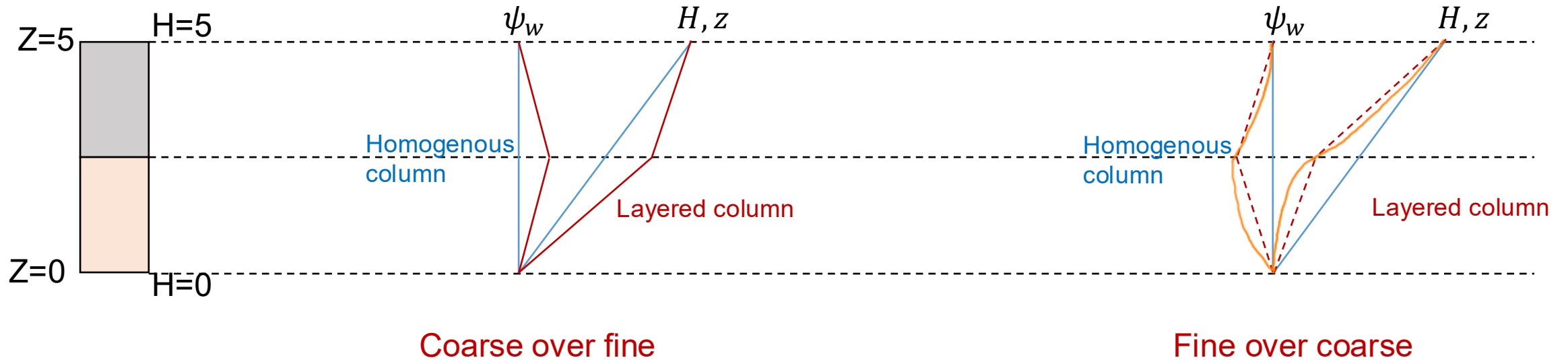
Horizontal flow (Homogeneous column)



- Unsaturated flow involves nonlinearities that make their behaviors differ from that of the saturated flow

Steady-State Unsaturated Flow

Vertical flow (in layered columns)

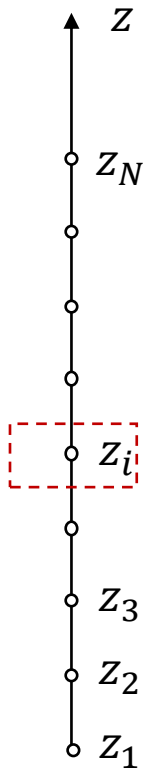


- Hydraulic head and pressure head are both continuous in space.
- Is water saturation continuous?

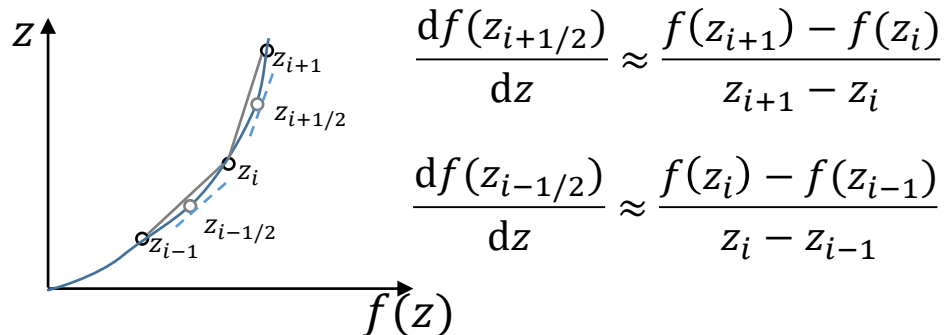
Steady-State Unsaturated Flow: Numerical Soln.

$$\frac{\partial \theta_w}{\partial t} - \frac{\partial}{\partial z} \left(K \frac{\partial \psi_w}{\partial z} \right) - \frac{\partial K}{\partial z} = 0 \quad \xrightarrow{\text{Steady-state}} \quad \frac{d}{dz} \left(K \frac{d\psi_w}{dz} \right) + \frac{dK}{dz} = 0 \quad (1)$$

- Equation (1) is a second-order ordinary differential equation in 1D.
- It is nonlinear because $K = K(\psi_w)$ is a nonlinear function.
- To solve it, we need two boundary conditions and we need to do it in an iterative procedure.

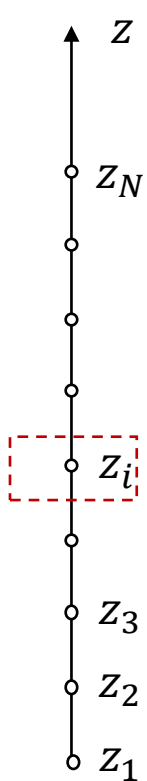


- How to solve this 1D **nonlinear ordinary differential equation**?
- Key idea: Divide the domain into many boxes and convert the **differential equation** to a system of nonlinear **algebraic equations**.
- Technique: Use finite difference to approximate derivatives



Steady-State Unsaturated Flow: Numerical Soln.

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$$\frac{d}{dz} \left(K \frac{d\psi_w}{dz} \right) \Big|_{z_i} \approx \frac{\left(K \frac{d\psi_w}{dz} \right)_{i+1/2} - \left(K \frac{d\psi_w}{dz} \right)_{i-1/2}}{\Delta z} \approx \frac{K_{i+1/2} \frac{\psi_{w,i+1} - \psi_{w,i}}{\Delta z} - K_{i-1/2} \frac{\psi_{w,i} - \psi_{w,i-1}}{\Delta z}}{\Delta z}$$

$$= \frac{K_{i+1/2}(\psi_{w,i+1} - \psi_{w,i}) - K_{i-1/2}(\psi_{w,i} - \psi_{w,i-1})}{\Delta z^2}$$

$$\frac{dK}{dz} \Big|_{z_i} \approx \frac{K_{i+1/2} - K_{i-1/2}}{\Delta z}$$

$$\Rightarrow \frac{K_{i+1/2}}{\Delta z^2} (\psi_{w,i+1} - \psi_{w,i}) - \frac{K_{i-1/2}}{\Delta z^2} (\psi_{w,i} - \psi_{w,i-1}) + K_{i+1/2} \frac{1}{\Delta z} - K_{i-1/2} \frac{1}{\Delta z} = 0$$

$$\Rightarrow \frac{K_{i+1/2}}{\Delta z^2} \psi_{w,i+1} - \left(\frac{K_{i+1/2}}{\Delta z^2} + \frac{K_{i-1/2}}{\Delta z^2} \right) \psi_{w,i} + \frac{K_{i-1/2}}{\Delta z^2} \psi_{w,i-1} + K_{i+1/2} \frac{1}{\Delta z} - K_{i-1/2} \frac{1}{\Delta z} = 0$$

This is an algebraic equation with 3 unknowns. We can write such an algebraic equation for each node or box and we can get N algebraic equations for the N unknowns $(\psi_{w,1}, \psi_{w,2}, \dots, \psi_{w,N})$

