

HWRS 561b: Physical Hydrogeology II

Two-phase flow and Richards equation (Part 1)

Agenda:

1. Mathematical Representation of Two-Phase Flow
2. Unsaturated Flow and the Richards' Equation

Reading: Chapter 11 (Pinder & Celia, 2006).

- ❖ Physical meaning of the SWC models (Brooks-Corey vs. van Genuchten)
- ❖ Specific yield and drainable porosity in the vadose zone
- ❖ Leverett J function/scaling and Miller-Miller scaling
- ❖ Unsaturated permeability (physical meaning) and mathematical description
- ❖ Extended Darcy's Law for unsaturated flow (Buckingham, 1907)

$$\mathbf{q}_w = - \frac{k_{r,w}(S_w) \mathbf{k}}{\mu_w} \nabla(p_w + \rho_w g z)$$

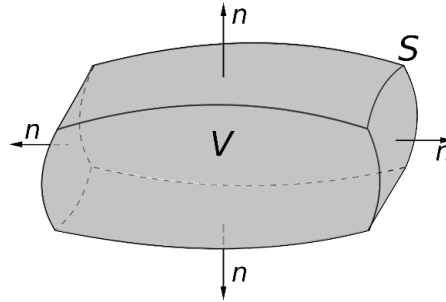
- ❖ How to derive unsaturated permeability from SWC?

Saturated flow

Derive the flow equation:

Recall: Divergence theorem

$$\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, ds = \int_{\Omega} \nabla \cdot \mathbf{q} \, dV$$



=> Divergence theorem converts a surface integral to a volume integral.

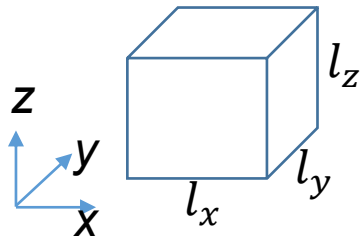
Mass conservation: Change of mass storage = mass in – mass out.

Rate of mass change:

$$\frac{d}{dt} \int_0^{l_x} \int_0^{l_y} \int_0^{l_z} \rho(x, y, z) \phi(x, y, z) \, dx \, dy \, dz = \frac{d}{dt} \int_{\Omega} \rho \phi \, dV$$

Net fluxes:

$$\begin{aligned} & \int_0^{l_y} \int_0^{l_z} q_x|_{x=0} \, dy \, dz - \int_0^{l_y} \int_0^{l_z} q_x|_{x=l_x} \, dy \, dz \\ & + \int_0^{l_x} \int_0^{l_z} q_y|_{y=0} \, dx \, dz - \int_0^{l_x} \int_0^{l_z} q_y|_{y=l_y} \, dx \, dz \\ & + \int_0^{l_x} \int_0^{l_y} q_z|_{z=0} \, dx \, dy - \int_0^{l_x} \int_0^{l_y} q_z|_{z=l_z} \, dx \, dy \end{aligned}$$



Saturated flow

$$\Rightarrow -\int_{\partial\Omega} \underline{q} \cdot \underline{n} dS = -\int_{\Omega} \nabla \cdot \underline{q} dV \quad \text{divergence theorem}$$

Thus, $\frac{d}{dt} \int_{\Omega} \rho\phi dV = -\int_{\Omega} \nabla \cdot \underline{q} dV$ $\left. \vphantom{\frac{d}{dt} \int_{\Omega} \rho\phi dV} \right\} \Omega \neq \Omega(t)$

$$\Rightarrow \int_{\Omega} \frac{d}{dt} (\rho\phi) dV = -\int_{\Omega} \nabla \cdot \underline{q} dV$$

$$\Rightarrow \int_{\Omega} \left[\frac{d}{dt} (\rho\phi) + \nabla \cdot \underline{q} \right] dV = 0$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial}{\partial t} (\rho\phi) + \nabla \cdot \underline{q} \right] dV = 0$$

$$\Rightarrow \frac{\partial}{\partial t} (\rho\phi) + \nabla \cdot \underline{q} = 0$$

Darcy's law: $\underline{q} = -\rho \underline{K} \nabla H$

\leftarrow Integral ≥ 0 , Ω is arbitrary \Rightarrow Integrand $= 0$

$$\frac{\partial}{\partial t} (\rho\phi) + \nabla \cdot (-\rho \underline{K} \nabla H) = 0$$

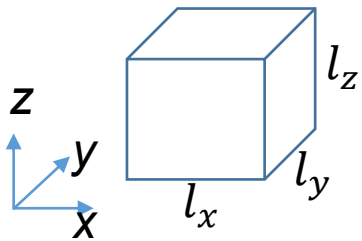
General 3D flow equation for a water-saturated porous medium.

Two-Phase Flow

Flux Law for two-phase flow (Muskat and Meres, 1936)

$$\begin{aligned}
 \mathbf{q}_w &= -\frac{k_{r,w}(S_w)\mathbf{k}}{\mu_w} \nabla(p_w + \rho_w g z) \\
 \mathbf{q}_{nw} &= -\frac{k_{r,nw}(S_w)\mathbf{k}}{\mu_{nw}} \nabla(p_{nw} + \rho_{nw} g z)
 \end{aligned}
 \left. \vphantom{\begin{aligned} \mathbf{q}_w \\ \mathbf{q}_{nw} \end{aligned}} \right\}
 \begin{aligned}
 \mathbf{q}_\alpha &= -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha} \nabla(p_\alpha + \rho_\alpha g z), \quad \alpha = nw \text{ or } w \\
 &= -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha} (\nabla p_\alpha - \rho_\alpha \mathbf{g}) \quad \nabla(\rho_\alpha g z) = -\rho_\alpha \mathbf{g}
 \end{aligned}$$

The derivation of the governing equations for two-phase flow shares the same procedure as that used for single-phase flow. The only difference is that a governing equation is needed for each fluid phase.



Rate of mass change:

$$\frac{\partial}{\partial t} \int_0^{l_x} \int_0^{l_y} \int_0^{l_z} \rho_\alpha(x,y,z,t) \phi(x,y,z,t) S_\alpha(x,y,z,t) dx dy dz = \frac{\partial}{\partial t} \int_\Omega \rho_\alpha \phi S_\alpha dV$$

Net fluxes:

$$\left. \begin{aligned}
 &\int_0^{l_y} \int_0^{l_z} \rho_\alpha \mathbf{q}_{\alpha,x} |_{x=0} dy dz - \int_0^{l_y} \int_0^{l_z} \rho_\alpha \mathbf{q}_{\alpha,x} |_{x=l_x} dy dz \\
 &\int_0^{l_x} \int_0^{l_z} \rho_\alpha \mathbf{q}_{\alpha,y} |_{y=0} dx dz - \int_0^{l_x} \int_0^{l_z} \rho_\alpha \mathbf{q}_{\alpha,y} |_{y=l_y} dx dz \\
 &\int_0^{l_x} \int_0^{l_y} \rho_\alpha \mathbf{q}_{\alpha,z} |_{z=0} dx dy - \int_0^{l_x} \int_0^{l_y} \rho_\alpha \mathbf{q}_{\alpha,z} |_{z=l_z} dx dy
 \end{aligned} \right\} \Rightarrow - \int_{\partial\Omega} \rho_\alpha \mathbf{q}_\alpha \cdot \mathbf{n} dS = - \int_\Omega \nabla \cdot (\rho_\alpha \mathbf{q}_\alpha) dV$$

Divergence theorem.

Two-Phase Flow

$$\Rightarrow \frac{\partial}{\partial t} \int_{\Omega} \rho_2 \phi \Omega_2 dV = - \int_{\Omega} \nabla \cdot (\rho_2 \underline{q}_2) dV$$

$\downarrow \Omega \neq \Omega(t)$

$$\Rightarrow \int_{\Omega} \frac{\partial}{\partial t} (\rho_2 \phi \Omega_2) dV = - \int_{\Omega} \nabla \cdot (\rho_2 \underline{q}_2) dV$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial}{\partial t} (\rho_2 \phi \Omega_2) + \nabla \cdot (\rho_2 \underline{q}_2) \right] dV = 0$$

$\downarrow \Omega$ is arbitrary

$$\Rightarrow \frac{\partial}{\partial t} (\rho_2 \phi \Omega_2) + \nabla \cdot (\rho_2 \underline{q}_2) = 0$$

$$\left. \begin{array}{l} \text{Flux law: } \underline{q}_2 = - \frac{k_{r,2} k}{\mu_2} \nabla (P_2 + \rho_2 g z) \end{array} \right\} \Rightarrow$$

$$\frac{\partial}{\partial t} (\rho_2 \phi \Omega_2) - \nabla \cdot \left[\rho_2 \frac{k_{r,2} k}{\mu_2} \nabla (P_2 + \rho_2 g z) \right] = 0$$

General two-phase flow governing equation.

Two-Phase Flow

The governing equations for the two fluid phases:

$$\text{Wetting phase} \quad \frac{\partial}{\partial t} (\rho_w \phi S_w) - \nabla \cdot \left[\rho_w \frac{k_{r,w} \mathbf{k}}{\mu_w} \nabla (p_w + \rho_w g z) \right] = 0 \quad (1)$$

$$\text{Nonwetting phase} \quad \frac{\partial}{\partial t} (\rho_{nw} \phi S_{nw}) - \nabla \cdot \left[\rho_{nw} \frac{k_{r,nw} \mathbf{k}}{\mu_{nw}} \nabla (p_{nw} + \rho_{nw} g z) \right] = 0 \quad (2)$$

Unknowns: $p_w, p_{nw}, S_w, S_{nw}, k_{r,w}, k_{r,nw}$ (Assuming $\rho_\alpha, \mu_\alpha, \phi, \mathbf{k}$ are known.)

=> 6 unknowns, 2 equations. We need 4 more equations to have a mathematically closed system.

$$S_w + S_{nw} = 1 \quad (3)$$

$$\left\{ \begin{array}{l} p_{nw} = p_w + p_c(S_w) \end{array} \right. \quad (4)$$

$$\left\{ \begin{array}{l} k_{r,w} = k_{r,w}(S_w) \end{array} \right. \quad (5)$$

$$\left\{ \begin{array}{l} k_{r,nw} = k_{r,nw}(S_w) \end{array} \right. \quad (6)$$

Constitutive equations
(e.g., VG or B-C models)

- 6 equations, 6 unknowns. The system of equations is mathematically closed.
- Substituting Eqs. (3-6) to Eqs. (1-2) => 2 equations and 2 unknowns.
- Several options for the primary variables. Two common options:
 - ✓ One phase pressure + one phase saturation (e.g., p_w, S_{nw})
 - ✓ Two phase pressures (e.g., p_w, p_{nw})

Unsaturated Flow: Richards' Equation

Let's now think about the air-water system in the vadose zone.

At a common condition: 1 atm, 20 °C

$$\rho_w \approx 1000 \text{ kg/m}^3$$

$$\rho_a \approx 1 \text{ kg/m}^3$$

$$\rho_a \ll \rho_w$$

$$\mu_w \approx 10^{-3} \text{ Pa} \cdot \text{s}$$

$$\mu_a \approx 1.8 \times 10^{-5} \text{ Pa} \cdot \text{s}$$

$$\mu_a \ll \mu_w$$

$$\text{Air flux: } \mathbf{q}_a = -\frac{k_{r,a} \mathbf{k}}{\mu_a} \nabla(p_a + \rho_a g z)$$

Richards' Assumptions

$\leftarrow \mu_a \ll \mu_w$

Air is essentially inviscid. In that sense, almost no gradient of energy potential is required to drive air flow.

$$\Rightarrow \nabla(p_a + \rho_a g z) \approx 0 \Rightarrow p_a + \rho_a g z \approx \text{const.}$$

At the ground surface ($z=0$), $p_a=0$

$$p_a(z) = 0 - \rho_a g z$$

$$\rho_a \approx 1 \text{ kg/m}^3$$

$\Rightarrow p_a \approx 0$ in the unsaturated zone.

$$\frac{\rho_a g z}{1 \times 10^5 \text{ Pa}} \approx \frac{1 \text{ kg/m}^3 \times 10 \text{ m/s}^2 \times z}{10 \times 100 \text{ m}} \ll 1 \text{ atm} \approx 10^5 \text{ Pa.}$$

Richards' Equation

Governing equations for air-water flow in porous media:

$p_a = 0$. Thus, no need to solve Equation (1)

$$\frac{\partial}{\partial t} (\rho_a \phi s_a) - \nabla \cdot \left[\frac{\rho_a k_{r,a} \mathbf{k}}{\mu_a} \nabla (p_a + \rho_a g z) \right] = 0 \quad (1)$$

$$\frac{\partial}{\partial t} (\rho_w \phi s_w) - \nabla \cdot \left[\frac{\rho_w k_{r,w} \mathbf{k}}{\mu_w} \nabla (p_w + \rho_w g z) \right] = 0 \quad (2)$$

$= 0 - p_c(s_w)$

$$s_a + s_w = 1 \quad (3)$$

$$k_{r,a} = k_{r,a}(s_w) \quad (4)$$

$$k_{r,w} = k_{r,w}(s_w) \quad (5)$$

$$p_a - p_w = p_c(s_w) \quad (6)$$

Now, this is the only eqn we need to solve. It's essentially the Richards' Eqn, but in a more general form.

If we simplify by assuming constant density and porosity:

$$\phi \frac{\partial s_w}{\partial t} - \nabla \cdot \left[\frac{k_{r,w} \mathbf{k}}{\mu_w} \nabla (p_w + \rho_w g z) \right] = 0$$

Richards' Eqn.