

# HWRS 561b: Physical Hydrogeology II

## Two-phase flow and Richards equation (Part 1)

### Agenda:

1. Mathematical Representation of Two-Phase Flow
2. Unsaturated Flow and the Richards' Equation

Reading: Chapter 11 (Pinder & Celia, 2006).

- ❖ Physical meaning of the SWC models (Brooks-Corey vs. van Genuchten)
- ❖ Specific yield and drainable porosity in the vadose zone
- ❖ Leverett J function/scaling and Miller-Miller scaling
- ❖ Unsaturated permeability (physical meaning) and mathematical description
- ❖ Extended Darcy's Law for unsaturated flow (Buckingham, 1907)

$$\mathbf{q}_w = - \frac{k_{r,w}(S_w) \mathbf{k}}{\mu_w} \nabla(p_w + \rho_w g z)$$

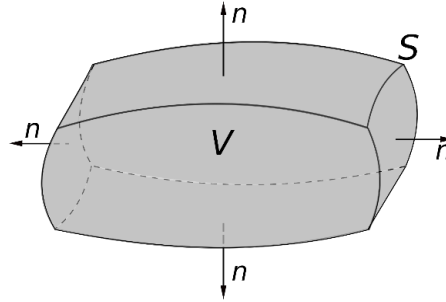
- ❖ How to derive unsaturated permeability from SWC?

# Saturated flow

Derive the flow equation:

Recall: Divergence theorem

$$\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, ds = \int_{\Omega} \nabla \cdot \mathbf{q} \, dV$$



=> Divergence theorem converts a surface integral to a volume integral.

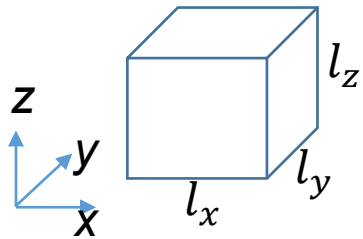
Mass conservation: Change of mass storage = mass in – mass out.

Rate of mass change:

$$\frac{d}{dt} \int_0^{l_x} \int_0^{l_y} \int_0^{l_z} \rho(x, y, z) \phi(x, y, z) \, dx \, dy \, dz = \frac{d}{dt} \int_{\Omega} \rho \phi \, dV$$

Net fluxes:

$$\begin{aligned} & \int_0^{l_y} \int_0^{l_z} q_x|_{x=0} \, dy \, dz - \int_0^{l_y} \int_0^{l_z} q_x|_{x=l_x} \, dy \, dz \\ & + \int_0^{l_x} \int_0^{l_z} q_y|_{y=0} \, dx \, dz - \int_0^{l_x} \int_0^{l_z} q_y|_{y=l_y} \, dx \, dz \\ & + \int_0^{l_x} \int_0^{l_y} q_z|_{z=0} \, dx \, dy - \int_0^{l_x} \int_0^{l_y} q_z|_{z=l_z} \, dx \, dy \end{aligned}$$



# Saturated flow

$$\Rightarrow -\int_{\partial\Omega} \underline{q} \cdot \underline{n} dS = -\int_{\Omega} \nabla \cdot \underline{q} dV \quad \text{divergence theorem}$$

Thus,  $\frac{d}{dt} \int_{\Omega} \rho\phi dV = -\int_{\Omega} \nabla \cdot \underline{q} dV$   $\left. \vphantom{\frac{d}{dt} \int_{\Omega} \rho\phi dV} \right\} \Omega \neq \Omega(t)$

$$\Rightarrow \int_{\Omega} \frac{d}{dt} (\rho\phi) dV = -\int_{\Omega} \nabla \cdot \underline{q} dV$$

$$\Rightarrow \int_{\Omega} \left[ \frac{d}{dt} (\rho\phi) + \nabla \cdot \underline{q} \right] dV = 0$$

$$\Rightarrow \int_{\Omega} \left[ \frac{\partial}{\partial t} (\rho\phi) + \nabla \cdot \underline{q} \right] dV = 0$$

$$\Rightarrow \frac{\partial}{\partial t} (\rho\phi) + \nabla \cdot \underline{q} = 0$$

Darcy's law:  $\underline{q} = -\rho \underline{K} \nabla H$

$\leftarrow$  Integral  $\geq 0$ ,  $\Omega$  is arbitrary  $\Rightarrow$  Integrand  $= 0$

$$\frac{\partial}{\partial t} (\rho\phi) + \nabla \cdot (-\rho \underline{K} \nabla H) = 0$$

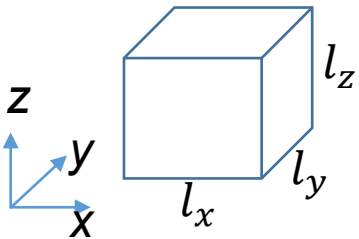
General 3D flow equation for a water-saturated porous medium.

# Two-Phase Flow

## Flux Law for two-phase flow (Muskat and Meres, 1936)

$$\left. \begin{aligned} \mathbf{q}_w &= -\frac{k_{r,w}(S_w)\mathbf{k}}{\mu_w} \nabla(p_w + \rho_w g z) \\ \mathbf{q}_{nw} &= -\frac{k_{r,nw}(S_w)\mathbf{k}}{\mu_{nw}} \nabla(p_{nw} + \rho_{nw} g z) \end{aligned} \right\} \begin{aligned} \mathbf{q}_\alpha &= -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha} \nabla(p_\alpha + \rho_\alpha g z), \quad \alpha = nw \text{ or } w \\ &= -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha} (\nabla p_\alpha - \rho_\alpha \mathbf{g}) \end{aligned} \quad \nabla(\rho_\alpha g z) = -\rho_\alpha \mathbf{g}$$

The derivation of the governing equations for two-phase flow shares the same procedure as that used for single-phase flow. The only difference is that a governing equation is needed for each fluid phase.



# Two-Phase Flow

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# Two-Phase Flow

The governing equations for the two fluid phases:

$$\text{Wetting phase} \quad \frac{\partial}{\partial t} (\rho_w \phi S_w) - \nabla \cdot \left[ \rho_w \frac{k_{r,w} \mathbf{k}}{\mu_w} \nabla (p_w + \rho_w g z) \right] = 0 \quad (1)$$

$$\text{Nonwetting phase} \quad \frac{\partial}{\partial t} (\rho_{nw} \phi S_{nw}) - \nabla \cdot \left[ \rho_{nw} \frac{k_{r,nw} \mathbf{k}}{\mu_{nw}} \nabla (p_{nw} + \rho_{nw} g z) \right] = 0 \quad (2)$$

Unknowns:  $p_w, p_{nw}, S_w, S_{nw}, k_{r,w}, k_{r,nw}$  (Assuming  $\rho_\alpha, \mu_\alpha, \phi, \mathbf{k}$  are known.)

=> 6 unknowns, 2 equations. We need 4 more equations to have a mathematically closed system.

$$S_w + S_{nw} = 1 \quad (3)$$

$$\left\{ \begin{array}{l} p_{nw} = p_w + p_c(S_w) \end{array} \right. \quad (4)$$

$$\left\{ \begin{array}{l} k_{r,w} = k_{r,w}(S_w) \end{array} \right. \quad (5)$$

$$\left\{ \begin{array}{l} k_{r,nw} = k_{r,nw}(S_w) \end{array} \right. \quad (6)$$

Constitutive equations  
(e.g., VG or B-C models)

- 6 equations, 6 unknowns. The system of equations is mathematically closed.
- Substituting Eqs. (3-6) to Eqs. (1-2) => 2 equations and 2 unknowns.
- Several options for the primary variables. Two common options:
  - ✓ One phase pressure + one phase saturation (e.g.,  $p_w, S_{nw}$ )
  - ✓ Two phase pressures (e.g.,  $p_w, p_{nw}$ )

# Unsaturated Flow: Richards' Equation

Let's now think about the air-water system in the vadose zone.

At a common condition: 1 atm, 20 °C

$$\rho_w \approx 1000 \text{ kg/m}^3$$

$$\rho_a \approx 1 \text{ kg/m}^3$$

$$\left. \begin{array}{l} \rho_w \approx 1000 \text{ kg/m}^3 \\ \rho_a \approx 1 \text{ kg/m}^3 \end{array} \right\} \rho_a \ll \rho_w$$

$$\mu_w \approx 10^{-3} \text{ Pa} \cdot \text{s}$$

$$\mu_a \approx 1.8 \times 10^{-5} \text{ Pa} \cdot \text{s}$$

$$\left. \begin{array}{l} \mu_w \approx 10^{-3} \text{ Pa} \cdot \text{s} \\ \mu_a \approx 1.8 \times 10^{-5} \text{ Pa} \cdot \text{s} \end{array} \right\} \mu_a \ll \mu_w$$

$$\text{Air flux: } \mathbf{q}_a = -\frac{k_{r,a} \mathbf{k}}{\mu_a} \nabla(p_a + \rho_a g z)$$

# Richards' Equation

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Governing equations for air-water flow in porous media:

$$\frac{\partial}{\partial t} (\rho_a \phi s_a) - \nabla \cdot \left[ \frac{\rho_a k_{r,a} \mathbf{k}}{\mu_a} \nabla (p_a + \rho_a g z) \right] = 0 \quad (1)$$

$$\frac{\partial}{\partial t} (\rho_w \phi s_w) - \nabla \cdot \left[ \frac{\rho_w k_{r,w} \mathbf{k}}{\mu_w} \nabla (p_w + \rho_w g z) \right] = 0 \quad (2)$$

$$s_a + s_w = 1 \quad (3)$$

$$k_{r,a} = k_{r,a}(s_w) \quad (4)$$

$$k_{r,w} = k_{r,w}(s_w) \quad (5)$$

$$p_a - p_w = p_c(s_w) \quad (6)$$